

# Type-2 Fuzzy Alpha-cuts

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## **Abstract**

Systems that utilise type-2 fuzzy sets to handle uncertainty have not been implemented in real world applications unlike the astonishing number of applications involving standard fuzzy sets. The main reason behind this is the complex mathematical nature of type-2 fuzzy sets which is the source of two major problems. On one hand, it is difficult to mathematically manipulate type-2 fuzzy sets, and on the other, the computational cost of processing and performing operations using these sets is very high. Most of the current research carried out on type-2 fuzzy logic concentrates on finding mathematical means to overcome these obstacles. One way of accomplishing the first task is to develop a meaningful mathematical representation of type-2 fuzzy sets that allows functions and operations to be extended from well known mathematical forms to type-2 fuzzy sets. To this end, this thesis presents a novel alpha-cut representation theorem to be this meaningful mathematical representation. It is the decomposition of a type-2 fuzzy set in to a number of classical sets. The alpha-cut representation theorem is the main contribution of this thesis.

This dissertation also presents a methodology to allow functions and operations to be extended directly from classical sets to type-2 fuzzy sets. A novel alpha-cut extension principle is presented in this thesis and used to define uncertainty measures and arithmetic operations for type-2 fuzzy sets. Throughout this investigation, a plethora of concepts and definitions have been developed for the first time in order to make the manipulation of type-2 fuzzy sets a simple and straight forward task. Worked examples are used to demonstrate the usefulness of these theorems and methods.

Finally, the crisp alpha-cuts of this fundamental decomposition theorem are by definition independent of each other. This dissertation shows that operations on type-2 fuzzy sets using the alpha-cut extension principle can be processed in parallel. This feature is found to be extremely powerful, especially if performing computation on the massively parallel graphical processing units. This thesis explores this capability and shows through different experiments the achievement of significant reduction in processing time.

# Summary of Notations

Table 1. Notations used in the thesis.

Notation	Abbreviation	Meaning
$U$		The unit interval $[0, 1]$ . It is the universe of membership grades in fuzzy sets, interval valued fuzzy sets, and of primary membership grades in type-2 fuzzy sets
$u_x$	PG	A membership grade in $U$ at domain value $x$ . It is also called the primary grade of a type-2 fuzzy set
$A$	FS	A fuzzy set including its special cases, crisp sets and intervals
$\hat{A}$	IVFS	An interval valued fuzzy set
$\underline{A}$	LMF	A lower membership function of an interval valued fuzzy set $\hat{A}$
$\bar{A}$	UMF	An upper membership function of an interval valued fuzzy set $\hat{A}$
$\tilde{A}$	T2FS	A type-2 fuzzy set
$A(x)$	MG	A fuzzy set membership grade of domain value $x$
$\hat{A}(x)$	MG	An interval valued fuzzy set membership grade of domain value $x$
$\tilde{A}(x), \tilde{A}_x$	VS	A type-2 fuzzy set membership grade of domain value $x$ . Also called a vertical slice
$J_x$	PM	A primary membership at domain value $x$
$\tilde{U}$		The unit interval $[0, 1]$ . It is the universe of secondary grades
$\tilde{u}_x$	SG	A secondary grade at domain value $x$ and associated with primary grade $u_x$
$A_e$	EFS	An embedded fuzzy set of type-2 fuzzy set $\tilde{A}$
$\tilde{A}_e$	ET2FS	An embedded type-2 fuzzy set of type-2 fuzzy set $\tilde{A}$
$FOU(\tilde{A})$	FOU	The foot print of uncertainty of type-2 fuzzy set $\tilde{A}$
$PS(\tilde{A})$	PS	The principal set of type-2 fuzzy set $\tilde{A}$
$PMF(\tilde{A})$	PMF	The principal membership function of type-2 fuzzy set $\tilde{A}$
$A_\alpha$	$\alpha$ -cut	An $\alpha$ -cut of fuzzy set $A$
$\hat{A}_\alpha$	$\alpha$ -cut	An $\alpha$ -cut of interval valued fuzzy set $\hat{A}$
$\tilde{A}_{\tilde{\alpha}}$	$\alpha$ -plane	An $\alpha$ -plane of type2 fuzzy set $\tilde{A}$

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# Chapter 1

## Introduction

The solution of real world problems have been the main goal of scientific research in general and engineering and computing research in particular. These problems are characterised by different kinds of uncertainty in which finding a solution should be able to accommodate. Probability theory and fuzzy logic are amongst the most prominent fields of handling uncertainties in such systems. This dissertation is concerned with uncertainties involving human perceptions of which fuzzy sets and systems are affiliated to.

Lately, various extensions for fuzzy logic have been developed to handle extra-levels of uncertainty in which the standard fuzzy logic fails to fully accommodate. Such extra-levels are inherent in many real world systems. Type-2 fuzzy sets are unique and conceptually appealing, because they are fuzzy extension rather than crisp. This dissertation concentrates on type-2 fuzzy sets as means of modelling uncertainties in the real world.

The beginning of the field of fuzzy logic and its explosive growth all took off from redefining the concept of a set. Zadeh (1965) used, *fuzzy sets*, to be the title of the seminal paper of this emergent field. Zadeh (1983) explained that the motivation behind such a move away from classical set theory is to be able to capture uncertainties inherent in human language. Later Zadeh (1996a) advocated fuzzy logic as a mechanism for computing with words, and for the manipulation of perceptions (Zadeh 2001). The main contribution of fuzzy sets is rooted in the two concepts of graduation and granulation (Zadeh 1996b, Zadeh 1997b). The ability to categorise (granulate) objects by a soft constraint of gradual membership rather than a hard binary constraint is what Klir & Yuan (1995) considered to be a paradigm shift.

Throughout the past four decades the field of fuzzy logic enjoyed active research in theory and applications. In fact, new emergent fields included fuzzy logic as one of its pillars, such as computational intelligence introduced by Bezdek (1994) and soft computing introduced by Zadeh (1997a). Extensions to standard fuzzy sets have been developed, such as fuzzy sets with interval membership grades (Zadeh 1975a), fuzzy sets with membership grades and non-membership grades (Atanassov 1986), fuzzy sets with lattice valued membership grades (Goguen 1969), and

fuzzy sets with different fuzzy levels (Gottwald 1979). Other concepts that are related to fuzzy sets have also been defined such as rough sets (Pawlak 1985), vague sets (Gau & Buehrer 2002), gray sets (Deng 1989), random sets (Goutsias et al. 1997), and bi-polar fuzzy sets (Dubois & Prade 2008).

Many fields of research have been developed that include the fuzzification of mathematical concepts. One can find from the dawn of fuzzy set theory the fuzzification of some fields such as control, systems, numbers and mathematics. The result of this fuzzification process produced new fields such as fuzzy control, fuzzy systems, fuzzy numbers, and fuzzy mathematics (Klir & Folger 1988, Dubois & Prade 1983, Zimmermann 2001). One of the important concepts being fuzzified is the concept of a fuzzy set itself. This have been proposed by Zadeh (1975a), since a fuzzy set is characterised by crisp membership grades, instead these fuzzy sets can be characterised by fuzzy membership grades. Zadeh called this a type-2 fuzzy set, it is a fuzzy set with membership grades represented as fuzzy sets themselves. The motivation behind this idea was to aid the development of approximate reasoning systems based on linguistic variables. It is no coincidence that type-2 fuzzy sets, linguistic variables, the extension principle and  $\alpha$ -cuts emerged in the same trilogy (Zadeh 1975a, Zadeh 1975b, Zadeh 1975c).

The following twenty years saw lack of interest in type-2 fuzzy logic and type-2 fuzzy sets. Only in the past fifteen years, type-2 fuzzy sets regained interest by researchers and serious research activity took place (John & Coupland 2007). Mendel (2001) justified the lack of interest to the lack of computational power at the time to be able to process such complex sets. Another justification advocated by John (1998) is the lack of applications that require the uncertainty preserved by the third dimension. John argued that since fuzzy sets are well known to be able deal with words represented by the concept of a linguistic variable, perceptions on the other have not been investigated. John (2000) showed that such applications require type-2 fuzzy sets, and some applications appeared to support this theme (John et al. 2000, Innocent & John 2004, John & Innocent 2005). The above mentioned implied that the research on type-2 fuzzy sets took two different but related directions, one that concentrate on finding methods that reduce computational complexity, and the other concentrate on finding meaningful problems that require the complex nature of type-2 fuzzy sets for the solution.

Researchers in type-2 fuzzy logic who concentrate on solving the computational complexity problem, mainly developed methods and applications for a special case of type-2 fuzzy sets those of interval membership grades (Mendel 2007). Only very recently type-2 fuzzy sets have been considered in test applications, in which it produced reasonable performance (Coupland & John 2007, Liu 2008, Wagner & Hagra 2010).

Uncertainty, on the other hand, plays an important role in fuzzy logic theory and applications. The amount of uncertainty inherent in a fuzzy set has been quantified using different methods. Klir (2006) presented the foundation for a research project that unifies the theories dealing with

uncertainty. Probability theory and fuzzy logic are central to Klir's research project which is called Generalised Information Theory. In this thesis type-2 fuzzy sets are utilised in the field of generalised information theory. The contribution presented in this thesis pave the way for active research on the relationship between type-2 fuzzy sets and different measures of uncertainty.

Type-2 fuzzy sets have been represented mathematically through five representations, namely, Mizumoto and Tanaka's representation, the vertical slice representation, the wavy slice representation, the geometric representation and the  $\alpha$ -plane/zSlice representation. In general all of them fail to define a justified mathematical formalism with a generalised calculus. This thesis provides such a formalism for the first time. Mizumoto and Tanaka's representation is first described by Mizumoto & Tanaka (1976) and has not been influential for applications, even though Mendel (2007) perceived it to be the starting point of all following theoretical investigation. In reality, this is not the representation developed by Zadeh (1975a) when first defined the concept of type-2 fuzzy sets. The importance of this representation stems from the definition of the extension principle for type-2 fuzzy sets which is defined by Mizumoto & Tanaka (1976). The second representation is called the vertical slice representation. It represents the original definition of type-2 fuzzy sets presented by Zadeh (1975a). Vertical slices are explained in Chapter 2 to reflect the meaning and motivation behind type-2 fuzzy sets. The vertical slice representation is neither used in many applications nor in theoretical investigations. It is useful for exploring and explaining type-2 fuzzy sets, as well as being the first step to visualise a solution to a given problem. The third is the wavy slice representation based on embedded type-1 and type-2 fuzzy sets (Mendel & John 2002). This representation is useful for theoretical investigation, especially for interval valued fuzzy sets, in which the structure of the solution can be determined immediately. It is the most computationally expensive solution, as it involves the computation of numerous embedded sets which requires astronomical number of calculations. The fourth representation is the geometric representation developed by Coupland & John (2007) which uses geometric primitives to define type-2 fuzzy sets. Although, it has shown practical potential does not have a closed form formula for mathematical manipulation and hence has limited theoretical impact. The final representation is the  $\alpha$ -plane or zSlice representation of type-2 fuzzy sets developed by Liu (2008) or Wagner & Hagrass (2008) respectively. This representation decomposes type-2 fuzzy sets into a collection of interval valued fuzzy sets. Although these two representations are developed independently, the reason behind their development is the same, i.e., to calculate the centroid of a type-2 fuzzy set. Interestingly, some hints to this representation can be found in the literature, and the first implied definition to this representation trace back to Zadeh (1975a). The intersection of two type-2 fuzzy sets has been defined through a similar method. Till this thesis, no extension principle have been defined for using the  $\alpha$ -plane/zSlice representation. This thesis provides an in depth investigation on the  $\alpha$ -plane representation theorem and its use in applications, and one of the main contributions of this thesis is the  $\alpha$ -plane extension principle presented in Chapter 3.

Despite the availability of several representations for type-2 fuzzy sets, still type-2 fuzzy sets are not fully developed as a theoretical alternative in the investigation of uncertainty-based information theory. This can be clearly seen in the research blue print presented by Klir (2006). The reason behind that is the lack of a mathematical formalism that allows simple manipulation of type-2 fuzzy sets. Moreover, the applications of type-2 fuzzy sets in real world situations are not fully accepted as a practical alternative. To this end, the novel representation presented in this thesis is proposed to serve type-2 fuzzy sets both theoretically and practically.

In summary, this thesis presents a novel decomposition theorem along with novel methods for manipulating type-2 fuzzy sets. These novel concepts allow meaningful theoretical investigation into type-2 fuzzy sets to take place. Additionally, these novel methods permit significant reduction in processing applications by allowing operations on type-2 fuzzy sets be processed in parallel. The novel alpha-cut representation theorem also opens type-2 fuzzy sets to the world of uncertainty-based information theories. This thesis is an in depth investigation into the building blocks of type-2 fuzzy sets hoping that the reader will be able to find type-2 fuzzy sets less complex and much easier to comprehend.

The findings and contributions presented in this thesis are shown to have great impact on both aforementioned research themes of type-2 fuzzy sets. The impact on theory and applications of type-2 fuzzy sets not only withhold to the confines of type-2 fuzzy sets, but spin out to affect the more general theory and applications of fuzzy sets.

## 1.1 Motivation

The problem of reasoning, computing, or engineering in real world environments requires the ability to work in environments characterised with uncertainty. This is indeed a difficult task which has been the subject of research for decades. Many methods and theories have been developed to capture different forms of uncertainty in which probability theory and fuzzy logic play great roles.

Fuzzy logic has been for a long time advocated to solve problems in a human-like manner. This stems from the concept of a linguistic variable which is extensively used in control applications. To be able to accommodate extra-levels of uncertainty, many extension to fuzzy logic have been developed. One of the motivations behind choosing type-2 fuzzy sets as a candidate for solving real world applications is its ability to capture uncertainties about the fuzzy set itself.

The choice of type-2 fuzzy logic is backed by very interesting statements such as, “*type-2 fuzzy logic model more uncertainty than type-1 fuzzy logic*”. This statement raises many questions one of the which is why is type-2 fuzzy logic not utilised in applications?

Other conclusions suggest that type-2 fuzzy systems outperformed type-1 fuzzy systems. Although it is a very contentious point, but a consensus about the extra degrees of freedom a type-2 fuzzy set provide by definition supports such claims. Consequently, some other questions emerge

such as, can the uncertainty of type-2 fuzzy sets be measured? and if so can it be compared to type-1 fuzzy sets?

To be able to define measures of uncertainty for type-2 fuzzy sets means that a mathematical formulation and calculus should be available (Klir 1991). Mendel & John (2002) provided a starting point on the latest mathematical representations of type-2 fuzzy sets. Soon it was realised that formulating a mathematical model and a methodology for applying functions and operations that are simple and practical is needed.

In type-1 fuzzy sets this capability has been developed in a comprehensive manner. Unfortunately, in the context of type-2 fuzzy sets this was not available. This was the starting point of the research carried out on type-2 fuzzy sets. To summarise, *“focusing on the uncertainty inherent in type-2 fuzzy sets and its relationship with uncertainty theories led to the development of a novel mathematical representation and new calculus for manipulating these type-2 fuzzy sets”*.

## 1.2 Hypothesis and Objectives

The research hypothesis of this thesis can be stated as follows:

*“Type-2 fuzzy sets are capable of capturing uncertainty about membership grades of fuzzy sets. The ability to quantify the amount of uncertainty that a type-2 fuzzy set captures needs a meaningful mathematical representation. This mathematical representation can only be functional through an operational calculus for manipulation. On the other hand, this representation should be computationally inexpensive to be practically applicable. The alpha-cut representation theorem forms the basis for the meaningful mathematical representation, and the alpha-cut extension principle serves as the calculus for manipulating this representation. The independent nature of these alpha-cuts makes them suitable for parallel processing.”*

This research hypothesis can be translated into the following objectives:

- Theoretically, type-2 fuzzy sets have the capability to model uncertainty more than standard fuzzy sets. This statement means that measures of uncertainty must be defined for type-2 fuzzy sets comparable to those of standard fuzzy sets.
- The first step for a theory to be a functional uncertainty theory, is to have a representation that allow functions to be characterised by known axioms (Klir 2006). It is argued in this thesis that this assertion is possible through a mathematical representation that preserves the relationship between the different set theoretic formalisms, i.e., type-2 fuzzy sets, interval valued fuzzy sets, standard fuzzy sets and classical sets. This means the first step is to define



a mathematical representation for type-2 fuzzy sets that overcomes the shortcomings of the current available representations.

- The second step in measuring the uncertainty in type-2 fuzzy sets, is to develop a calculus to deal with these uncertainty functions (Klir 2006). It is also argued that this is possible by developing a method for extending operations from a formalism that is already capable of dealing with these uncertainty functions to the new formalism.
- A justification for defining each measure of uncertainty has to be defined, such justification is essential for giving a meaning to the amount of uncertainty measured (Klir 2006). Here an argument is followed that since the operations and functions are preserved through an extension process, by definition it preserves the justification and meaning of the measure.
- Practically, type-2 fuzzy sets are complex by nature and their logic is characterised by expensive computational cost. Many representations are developed to overcome this obstacle, and still the applicability of type-2 fuzzy sets industrially is very limited. This means for the mathematical formulation should be simple to compute.
- One of the popularly growing methods for reducing computational time is parallel processing. The ability to break a problem in to several independent structures is a major research problem for many fields and disciplines. It is argued in this thesis that the newly developed models and calculi presented throughout this thesis are parallel in nature, and this capability could be exploited in order to provide significantly lower computational costs.

### **1.3 Contributions of the Thesis**

To summarise this thesis contribution to the advancement of the field of type-2 fuzzy sets through the following:

1. Novel method of decomposing type-2 fuzzy sets into classical sets.
2. Novel method of extending operations from crisp sets to type-2 fuzzy sets.
3. Novel method of extending operations from interval valued fuzzy sets to type-2 fuzzy sets.
4. New uncertainty measures for type-2 fuzzy sets using the novel methods above.
5. New arithmetic operations for type-2 fuzzy sets using the novel methods above.
6. Parallel processing operations on type-2 fuzzy sets for the first time.

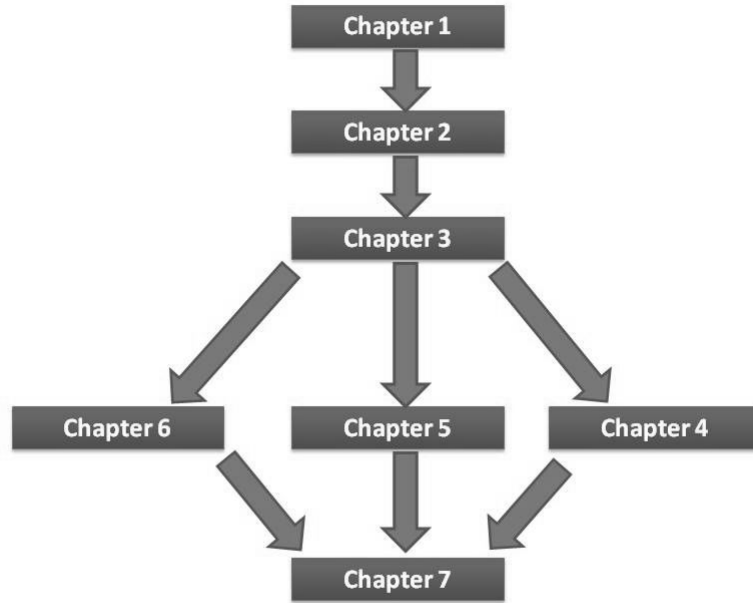


Fig. 1.1. The dependencies between chapters.

## 1.4 Structure of the Thesis

The thesis is organised as follows:

- Chapter 2 reviews the necessary notation and mathematical concepts essential for the development of the novel methods of the thesis. It starts by reviewing the literature on classical fuzzy set definitions, then interval valued fuzzy sets, and finally type-2 fuzzy sets. This chapter forms the background material leading to the development of the alpha-cut representation theorem of Chapter 3.
- In Chapter 3 the novel alpha-cut representation theorem of type-2 fuzzy sets is developed which allow type-2 fuzzy sets to be defined using a collection of classical sets or intervals. This representation theorem is the main contribution of this thesis. This chapter also presents a generalised alpha-cut extension principle capable of extending a wide range of functions, operations and relations from classical sets to type-2 fuzzy sets. Within this chapter all the required mathematical derivations and proofs are provided supported with many worked examples. Moreover, some essential operations have been defined in this chapter such as the core, support, and containment of type-2 fuzzy sets. Furthermore, interval valued fuzzy

sets have also been utilised in this chapter and a generalised alpha-plane extension principle have also been defined utilising the connection between type-2 fuzzy sets and interval valued fuzzy sets shown in Chapter 2. Finally, some essential concepts such as the reduction rule and the cutworthy property condition has been defined in order to aid in theoretical studies of type-2 fuzzy sets. Chapter 3 is the heart of this thesis and constitutes the main topics, concepts and theorems of this dissertation. The following chapter makes use of these concepts to define measures of uncertainty for type-2 fuzzy sets.

- Chapter 4 defines new measures of uncertainty extended to type-2 fuzzy sets using the novel methods presented in Chapter 3. These uncertainty measures are very important in comparing different aspects of type-2 fuzzy sets. The uncertainty measures chosen are the most common measures used in applications of fuzzy sets namely, cardinality, similarity, subsethood, fuzziness, and non-specificity.
- In Chapter 5, the second class of functions and operations are defined, these are arithmetic operations. A complete theory of type-2 fuzzy numbers and their arithmetic operations are developed using methods defined in Chapter 3.
- Chapter 6 present experiments which are conducted to show the potential of the novel representation and its extension principle in reducing computational complexity through the highly parallel structure of the representation. GPU computing is used because of its high speed in processing parallel structures. It is the first time for a type-2 fuzzy set operation to be implemented on a GPU. The experiments and the results are reported in Chapter 6.
- Chapter 7 is the conclusion of the thesis. Towards the end of this thesis many research questions are put forward for further research.

The flow and relationship between the chapters of dissertation can be seen in Figure 1.1. Throughout this journey a plethora of concepts, methods and mathematical formulations are defined and explained with worked examples and figures. It has to be admitted that this research project started with many questions and despite of the many novel contributions presented in this thesis, it ended with even more questions to be answered.

## Chapter 2

# Representations of Type-2 Fuzzy Sets

The work presented in this chapter presents the foundations necessary to understand the novel type-2 fuzzy set representation and definitions presented in the following chapter and the novel methodology for extending operations presented later in the thesis. Type-2 fuzzy sets have widely been accepted as capable of modeling higher orders of uncertainty than type-1 fuzzy sets (Mendel 2007, John & Coupland 2007, John 1998). This assertion has been the driving force behind much of the advancement of type-2 fuzzy set theories and applications (Mendel 2007). One of the important aspects of such advancement, if not the most important aspect, is finding a suitable representation that is theoretically meaningful and practically tractable.

By being theoretically meaningful, the representation should allow mathematical operations to be derived naturally and should have interpretable solutions. To this end, the available representations vary in their contribution, and this chapter discusses the role of these representations in theoretical understanding and mathematical manipulation.

For a representation to be practically tractable means, it should have the ability to reduce the computational cost in order to allow practitioners use type-2 fuzzy sets in their applications. To this end, the representations available are competing to be widely applied in industrial solutions. Some representations achieve one of these goals over the other and ultimately both goals are sought to be achieved.

Fuzzy logic features “*a tolerance for imprecision which can be exploited to achieve tractability, robustness, low solution cost and better rapport with reality*” (Zadeh 1996a). This crucial feature allowed many applications to be implemented using standard fuzzy logic, which type-2 fuzzy logic has yet to do so. This is the question behind much of the research conducted on type-2 fuzzy sets, i.e., how can type-2 fuzzy logic handle the trade-off between precision and computational cost? This chapter serves as a starter for this discussion by reviewing the relevant type-2 fuzzy set literature. In this chapter type-1 fuzzy sets, interval valued fuzzy sets and type-2 fuzzy sets are discussed mainly from the mathematical formulation perspective. All necessary mathematical conventions, notations, and definitions that are used throughout the thesis are defined

within this chapter.

This chapter is a literature review chapter that surveys most of the mathematical foundations for the rest of the thesis, moreover, it provides a sound contribution by unifying the many scattered notations found in the type-2 fuzzy logic community and directly relates it to relative type-1 fuzzy logic and interval valued fuzzy logic standard mathematical notations.

## 2.1 Type-1 Fuzzy Sets

This chapter begins with a discussion of the mathematical definitions of type-1 fuzzy sets (**T1FS**). Throughout the thesis a T1FS is sometimes referred to as a fuzzy set (**FS**) for short, and the two terms are used interchangeably. Zadeh (1965) defined fuzzy sets to be sets containing elements that belong to the sets with grades of membership rather than the elements being members of the set or not. Let,  $x \in X$ , be an element of the universe  $X$  that belongs to the fuzzy set,  $A \in X$ . The grade of membership,  $u_x$ , associated with each element,  $x$ , is a value in the unit interval,  $U = [0, 1]$ , i.e.  $u_x \in U$ . In general, a fuzzy set can be represented as the union of sets containing ordered pairs associating discrete points of the domain with their corresponding membership grades, i.e.,

$$\begin{aligned} A &= \{(x, u_x) \mid \forall x \in X, u_x \in U\} \\ &= \int_{\forall x \in X} u_x / x, u_x \in U \\ &= \bigcup_{\forall x \in X} \{(x, u_x) \mid u_x \in U\} \end{aligned} \tag{2.1}$$

In the second line of this equation, a classical fuzzy notation that involves the integration symbol  $\int$  and the division  $/$  is used. The integration represents the union and not standard integration and the division represents association and not standard division. Throughout this thesis it is believed that this standard fuzzy notation is confusing and will not be used, instead, standard mathematical notations are used (e.g.  $\cup$  for set union). Since each element in a fuzzy set is assigned a distinct membership grade (i.e. one and only one membership grade), then it is sensible and convenient to express these sets as functions. In the literature two notations have been widely used to denote fuzzy sets. The older and yet most widely used notation does not consider fuzzy sets as functions and rather define a function that acts on the set and returns a membership grade. The most common symbol,  $\mu_A$ , is used to denote the membership function (**MF**) of the fuzzy set,  $A$ . In this view, the symbol of the actual set is distinguished from the symbol of its MF. In the second notation, the symbol used for the fuzzy set is the same as the one used for the MF, i.e.  $A$ , actually the fuzzy set itself is a function. The fact that each fuzzy set is completely and uniquely defined by one particular MF justifies the double use of the same symbol (Klir & Yuan 1995). In this thesis, the second notation is used, for example, for the fuzzy set,  $A$ , the membership grade of an element,  $x$ ,

is  $A(x)$ . Formally, the following functional definition is adapted\* in this thesis.

**Definition 2.1.1 (Fuzzy Sets)** *A fuzzy set,  $A$ , defined over universe  $X$  is a function defined as follows:*

$$\begin{aligned} A : X &\rightarrow U \\ x &\mapsto u_x \end{aligned}$$

where  $x \in X$ ,  $U = [0, 1]$  is the unit interval, and  $u_x \in U$  is the membership grade of element  $x$  in FS  $A$ .

*Adapted from Zadeh (1965)*

Then,  $F(X)$ , is defined to be the set of all FSs in universe  $X$ . This definition is saying that  $A$  is a function from domain  $X$  to co-domain  $U$ , and each element  $x \in X$  of the domain maps to element  $u_x \in U$  of the co-domain, which evidently means that  $A(x) = u_x$ . This functional definition can also be used to re-express Eq. 2.1 as follows:

$$\begin{aligned} A &= \{(x, A(x)) \mid \forall x \in X, A(x) \in U\} \\ &= \bigcup_{\forall x \in X} (x, A(x)), A(x) \in U \end{aligned} \tag{2.2}$$

Zadeh established the connection between FSs and classical sets, called crisp sets hereafter, and showed that FSs are generalisations of classical sets by definition. Crisp sets are characterised by functions that return only one of two values: 1, if the element belongs to the set and, 0, if the element does not. This function is classically known as the characteristic function. In essence, a characteristic function is a MF and a crisp set is a special fuzzy set.

**Definition 2.1.2 (Crisp Sets)** *A crisp set,  $A^\dagger$ , over  $X$  is a function defined as follows:*

$$\begin{aligned} A : X &\rightarrow \{0, 1\} \\ x &\mapsto u_x \end{aligned} \tag{2.3}$$

Then,  $C(X)$ , is defined to be the set of all crisp sets on universe  $X$ . In this definition the membership grade  $A(x) = u_x \in \{0, 1\}$  still takes values in the unit interval, but only special values zero or one. Another special fuzzy set is an interval (Moore & Lodwick 2003). In mathematics, interval analysis has gained much importance since the seminal publication by Moore (1966).

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\*Throughout the thesis whenever a definition is adapted from a reference, it means that the concept is slightly changed from the formation it appeared in its reference in order to be suitable for the notational conventions of this thesis.

<sup>†</sup>Since crisp sets are considered special fuzzy sets, the same notation of a fuzzy set is used for crisp sets.

**Definition 2.1.3 (Intervals)** An *interval*,  $A$ , over  $X$  is defined by  $A = [\underline{x}, \bar{x}]$  where  $\underline{x}, \bar{x} \in X$  and  $\underline{x} \leq \bar{x}$ .

*Adapted from Moore et al. (2009)*

Then,  $I(X)$ , is defined to be the set of all intervals on universe  $X$ . An interval  $A = [\underline{x}, \bar{x}] \in X$  is a restricted crisp set on a continuous domain. It can be seen as a relation between domain values  $x \in X$ , where their associated membership grades  $A(x)$  are 1 for all  $\underline{x} \leq x \leq \bar{x}$  and 0 otherwise. In Figure 2.1 a discrete fuzzy set  $A = \{(1, 0.4), (3, 1.0), (5, 0.6)\}$  and a continuous fuzzy set  $B$  having a triangular shape are represented in a 2D coordinates. Figure 2.2 shows a one dimensional scale of crisp set  $A = \{1, 3, 5\}$  and interval  $B = [2, 6]$ . It is clear that there is no need to introduce a second dimension to represent these sets since their membership is always at unity when their elements are part of the set. Figure 2.3, shows a visual realisation of the fact that crisp sets and intervals are special cases of fuzzy sets. Introducing the second dimension that represents fuzzy set membership grades is the key modification to Figure 2.2.

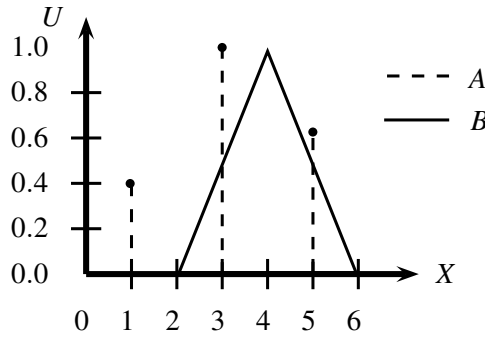


Fig. 2.1. Discrete fuzzy set  $A = \{(1, 0.4), (3, 1.0), (5, 0.6)\}$  and continuous triangular fuzzy set  $B$  represented in 2D coordinates.

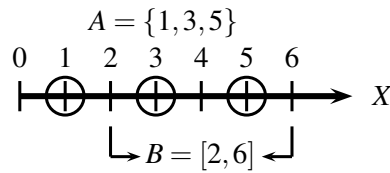


Fig. 2.2. 1D representation of crisp set  $A = \{1, 3, 5\}$  and interval  $B = [2, 6]$ .

In the mathematics of fuzzy sets two main operations are of most importance, the maximum and the minimum operations. These operations are introduced by Zadeh to define the union and intersection of fuzzy sets, and later widely used to define other operations as will be demonstrated later in this section.

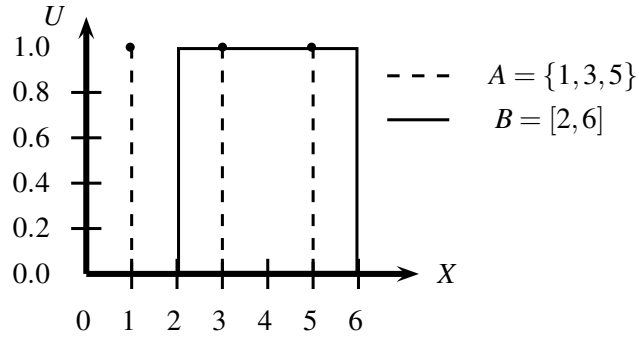


Fig. 2.3. 2D representation of crisp set  $A = \{1, 3, 5\}$  and interval  $B = [2, 6]$ .

**Definition 2.1.4** Let,  $a$  and  $b \in \mathfrak{R}$ , then,

$$a \vee b = \max(a, b)$$

$$a \wedge b = \min(a, b)$$

*Adapted from Zadeh (1965)*

These operations can be extended to intervals in a straight forward manner.

**Definition 2.1.5** Let,  $A = [\underline{a}, \bar{a}]$  and  $B = [\underline{b}, \bar{b}]$ , then,

$$\begin{aligned} A \vee B &= [\underline{a}, \bar{a}] \vee [\underline{b}, \bar{b}] \\ &= [\underline{a} \vee \underline{b}, \bar{a} \vee \bar{b}] \end{aligned}$$

$$\begin{aligned} A \wedge B &= [\underline{a}, \bar{a}] \wedge [\underline{b}, \bar{b}] \\ &= [\underline{a} \wedge \underline{b}, \bar{a} \wedge \bar{b}] \end{aligned}$$

*Adapted from Moore et al. (2009)*

Based on these basic operations, the union, intersection and complementation between fuzzy sets may be defined.

**Definition 2.1.6** Let,  $A$  and  $B \in F(X)$ , then,

$$(A \cup B)(x) = A(x) \vee B(x) \quad (\text{Union})$$

$$(A \cap B)(x) = A(x) \wedge B(x) \quad (\text{Intersection})$$

$$A'(x) = 1 - A(x) \quad (\text{Complementation})$$



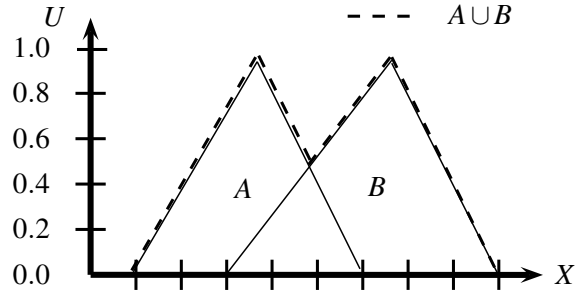


Fig. 2.4. Union of two fuzzy sets.

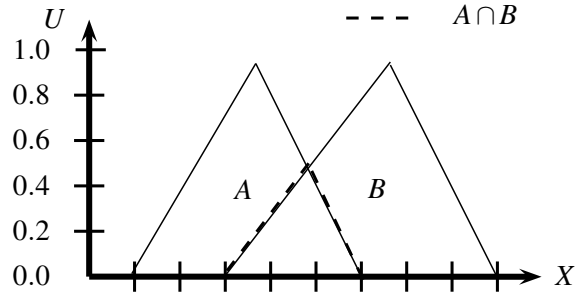


Fig. 2.5. Intersection of two fuzzy sets.

*Adapted from Zadeh (1965)*

These operations are called Zadeh's operations as they were first defined by Zadeh and the union is shown in Figure 2.4, intersection in Figure 2.5 and complementation in Figure 2.6. Different axiomatic systems (i.e. t-norms and t-conorms) have been developed to define operations that act in a similar manner (Mizumoto & Tanaka 1976, Mizumoto & Tanaka 1981). In this thesis only Zadeh's basic definitions are applied. Some commonly used terms and definitions are presented below.

**Definition 2.1.7 (FS Height)** *The height of a FS is the largest membership grade attained by any element in that set, i.e.,*

$$h(A) = \sup_{\forall x} A(x)$$

*where sup is the supremum.*

*Adapted from Zadeh (1965)*

**Definition 2.1.8 (FS Support)** *The support of a FS is the crisp set that contains all elements of,*

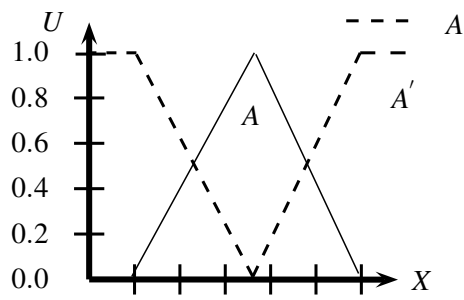


Fig. 2.6. Complement of a fuzzy set.

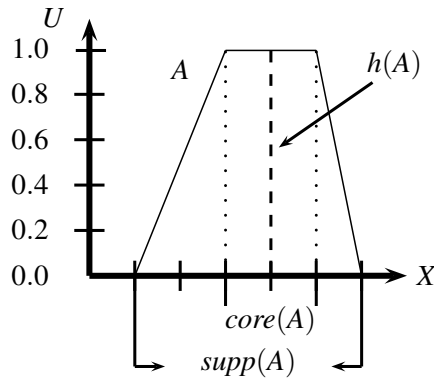


Fig. 2.7. Height, core, and support of Fuzzy Sets.

$A$ , that have nonzero membership grades, i.e.,

$$\text{supp}(A) = \{x \in X \mid A(x) > 0\}$$

*Adapted from Klir & Yuan (1995)*

**Definition 2.1.9 (FS Core)** The core of a FS is the crisp set that contains all elements of,  $A$ , that have membership grades equal to one, i.e.,

$$\text{core}(A) = \{x \in X \mid A(x) = 1\}$$

*Adapted from Klir & Yuan (1995)*

These three definitions are shown in Figure 2.7. Zadeh (1965) defines the core to be the set of domain values associated with membership grades equal to the height of that set. Since the height may be at unity as well, then Definition 2.1.9 is a special case of Zadeh's definition.

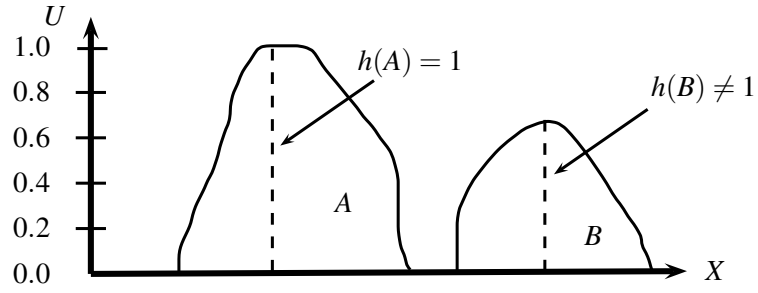


Fig. 2.8. Normal fuzzy set A and subnormal fuzzy set B.

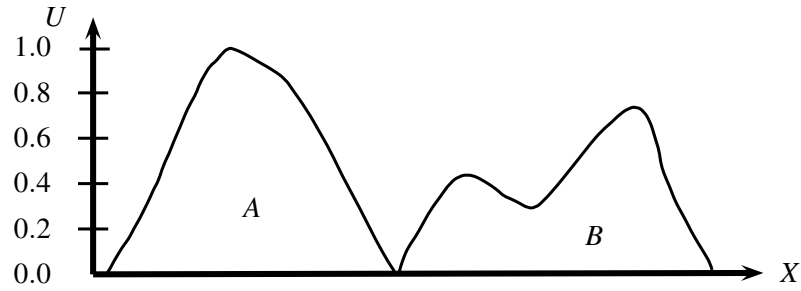


Fig. 2.9. Convex fuzzy set A and nonconvex fuzzy set B.

**Definition 2.1.10 (Normal FS)** A FS,  $A$ , is said to be normal if it has height equal to one, i.e.,  $h(A) = 1$ .

*Adapted from Klir & Yuan (1995)*

Any FS that is not normal is called subnormal FS. Figure 2.8 shows two fuzzy sets,  $A$  is normal and  $B$  is subnormal.

**Definition 2.1.11 (Convex FS)** Let  $A$  be a FS in a continuous domain,  $X$ . Then  $A$  is said to be convex if and only if for every pair of points  $x_i$  and  $x_j$  in  $A$  where  $i \neq j$ , the following inequality is satisfied:

$$A(\lambda x_i + (1 - \lambda)x_j) \geq \min(A(x_i), A(x_j)), \forall \lambda$$

where  $\lambda \in [0, 1]$ .

*Adapted from Zadeh (1965)*

The fuzzy set which is not convex are called non-convex FSs. Figure 2.9 shows two fuzzy sets,  $A$  is convex and  $B$  is non-convex.

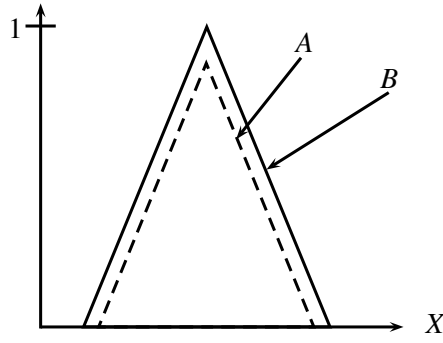


Fig. 2.10. Fuzzy set  $A$  is contained in fuzzy set  $B$ , i.e.,  $A \subseteq B$ .

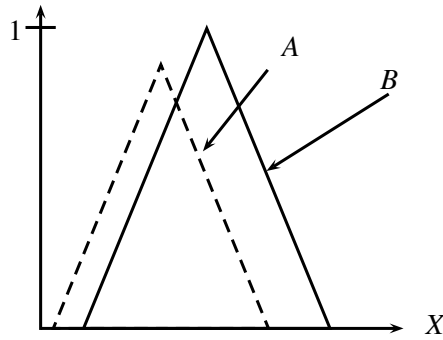


Fig. 2.11. Fuzzy set  $A$  is not contained in fuzzy set  $B$ .

**Definition 2.1.12 (FS Containment)** Let,  $A$  and  $B \in \mathcal{F}(X)$ , then,

$$A \subseteq B \Leftrightarrow A(x) \leq B(x), \forall x$$

*Adapted from Zadeh (1965)*

Figure 2.10 shows the containment between two fuzzy sets, and Figure 2.11 shows two fuzzy sets not contained in each other. This definition is a generalisation of the classical subthood between crisp sets and in some publications, e.g. (Zwick et al. 1987, Young 1996, Klir & Yuan 1995), the name subthood or inclusion is used for the same definition. Since there are many operations that are deemed to be subthood or inclusion operations the term containment is reserved in this thesis to distinguish this particular case of a fuzzy set being totally contained within another fuzzy set<sup>‡</sup>. Zadeh also defined the extension principle (**EP**) in order to extend operations and functions from crisp sets to fuzzy sets.

<sup>‡</sup>In Chapter 3 a discussion on fuzzy subthood will elaborate on this issue.

**Theorem 2.1.1 (FS EP)** Let,  $X = X_1 \times \dots \times X_n$ , be the Cartesian product of universes, and  $A_1, \dots, A_n$  be fuzzy sets in each universe respectively. Also let  $Y$  be another universe and  $B \in Y$  be a fuzzy set such that  $B = f(A_1, \dots, A_n)$ , where  $f : X \rightarrow Y$  is a monotonic mapping. Then Zadeh's extension principle (EP) is defined as follows:

$$B \Leftrightarrow B(y) = \sup_{(x_1, \dots, x_n) \in f^{-1}(y)} \min(A_1(x_1), \dots, A_n(x_n))$$

where  $f^{-1}(y)$  is the inverse function of  $y = f(x_1, \dots, x_n)$ .

*Adapted from Zadeh (1975a)*

The EP has been used extensively in the literature of fuzzy sets. It shares a resemblance to the *united extension* (Moore 1966) used to extend functions from reals to intervals in interval analysis.

### 2.1.1 Alpha-cuts of Fuzzy Sets

One of the most important concepts in fuzzy set theory and applications is the  $\alpha$ -cut decomposition theorem developed by Zadeh (1971) under the name *resolution identity*. These cuts are crisp sets associated with certain levels  $\alpha$ , that represent distinct grades of membership.

**Definition 2.1.13 (FS  $\alpha$ -cut)** An  $\alpha$ -cut of a FS,  $A$ , is a crisp set defined as follows:

$$A_\alpha = \{x \in X \mid A(x) \geq \alpha, \alpha \in [0, 1]\}$$

*Adapted from Zadeh (1973)*

What is interesting is the interpretation of these  $\alpha$ -cuts, they are classical sets that contain elements of the domain associated with membership grades greater than or equal to a certain level  $\alpha$ . In Figure 2.12 a continuous triangular fuzzy set  $A$  is shown with some  $\alpha$ -cuts, and in Figure 2.13 a discrete fuzzy set is shown. For example assume that a normal triangular FS, *Tall*, represented as the triple  $\langle 160, 180, 200 \rangle$ . Then its  $\alpha$ -cuts at level 0.5 is  $Tall_{0.5} = [170, 190]$ . This means that all domain values within this interval definitely belong to fuzzy set *tall* with membership grade 0.5. The idea behind the  $\alpha$ -cut representation is to define a useful special fuzzy set that is associated with each  $\alpha$ -cut. Klir & Yuan (1995) defined the associated fuzzy set by using an indicator function,  $I_{A_\alpha}$ , acting on  $x \in X$  such that,

$$I_{A_\alpha}(x) = \begin{cases} 1, & x \in A_\alpha \\ 0, & x \notin A_\alpha \end{cases}$$

Then a fuzzy set associated with each  $\alpha$ -cut,  $\alpha I_{A_\alpha}$ , is defined as follows:

$$\alpha I_{A_\alpha} = \{(x, \alpha I_{A_\alpha}(x)) \mid \forall x \in X, \alpha I_{A_\alpha}(x) = \alpha \cdot I_{A_\alpha}(x)\}$$

Alternatively, since  $A_\alpha$  is a crisp set then directly  $A_\alpha(x) \in \{0, 1\}$  and in a more direct approach the associated fuzzy set which is hereafter called  $\alpha$ -fuzzy set ( $\alpha$ -FS) is defined as follows:

**Definition 2.1.14 ( $\alpha$ -FS)** A special FS ( $\alpha$ -FS),  $\alpha A_\alpha \in F(X)$ , can be defined as follows:

$$\begin{aligned} \alpha A_\alpha &\Leftrightarrow \alpha A_\alpha(x) = \alpha \cdot A_\alpha(x) \text{ or alternatively} \\ &= \alpha \wedge A_\alpha(x) \end{aligned}$$

Figure 2.14 shows a continuous fuzzy sets and three  $\alpha$ -cuts. It also shows the corresponding  $\alpha$ -FSs for each  $\alpha$ -cut. Using this definition the  $\alpha$ -cut decomposition theorem can be defined in a straight forward manner.

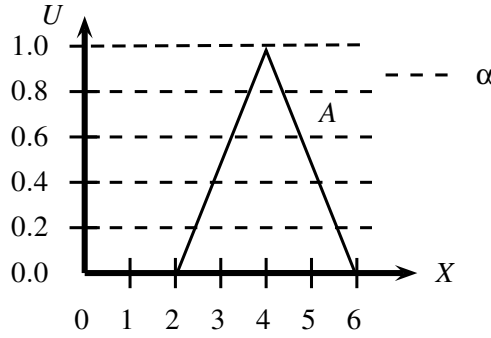


Fig. 2.12. The  $\alpha$ -cuts of continuous triangular fuzzy set A.

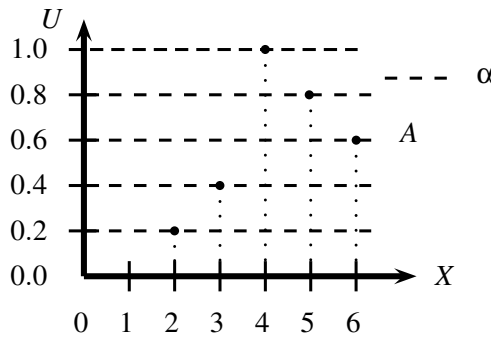


Fig. 2.13. The  $\alpha$ -cuts of discrete fuzzy set A.

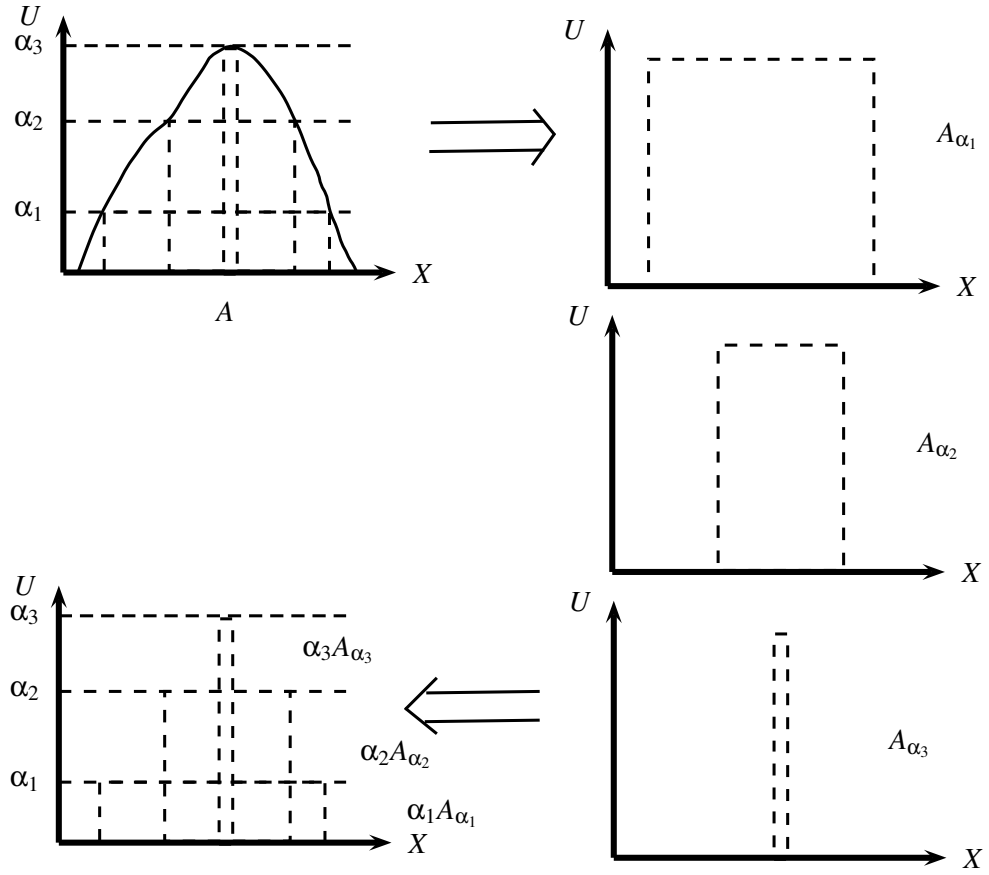


Fig. 2.14. The  $\alpha$ -FSs of fuzzy set  $A$ .

**Theorem 2.1.2 ( $\alpha$ -cut Representation Theorem)** A FS,  $A$ , can be represented (decomposed) by the union of all its  $\alpha$ -FSs, i.e.,

$$A = \bigcup_{\forall \alpha} \alpha A_{\alpha}$$

*Adapted from Zadeh (1975a)*

The membership grades of a fuzzy set can be regenerated from its decomposed  $\alpha$ -FSs directly.

**Corollary 2.1.1** *It is easy to check that the following holds:*

$$A(x) = \sup_{\forall \alpha} \alpha A_{\alpha}(x)$$

*Adapted from Nguyen (1978)*

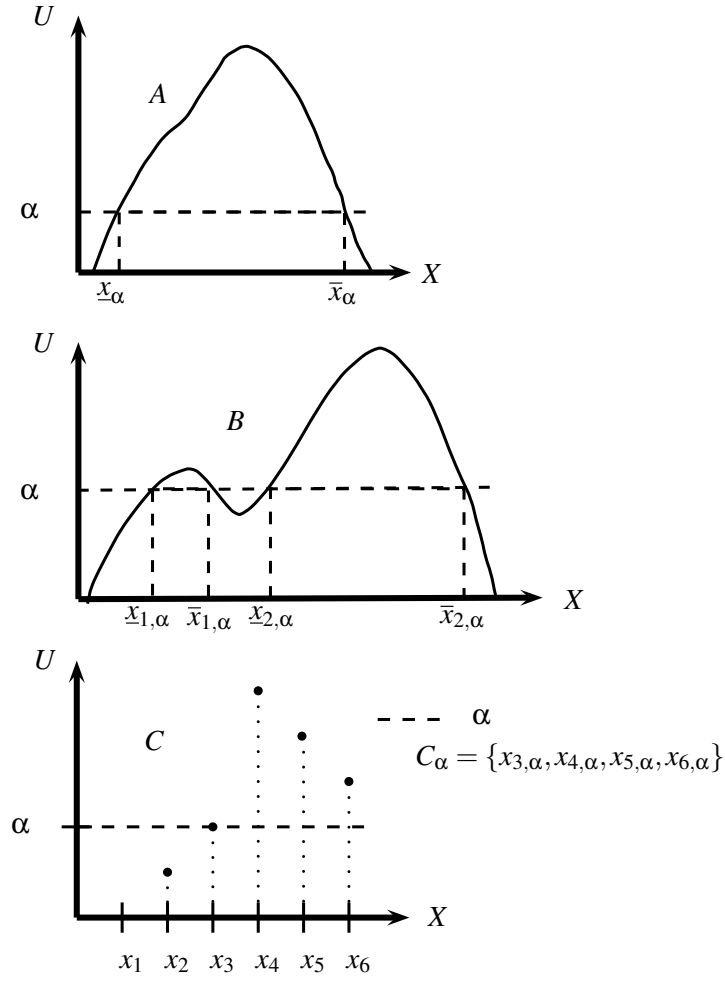


Fig. 2.15. The  $\alpha$ -cuts of continuous and convex FS  $A$ , nonconvex FS  $B$ , and discrete FS  $C$ .

Another less important variant of  $\alpha$ -cuts is the more strict version known as the strong  $\alpha$ -cut (Klir & Yuan 1995).

$$A_{\alpha^+} = \{x \in X \mid A(x) > \alpha, \alpha \in [0, 1]\} \quad (2.4)$$

The set that contains all the levels  $\alpha \in [0, 1]$  that represent distinct  $\alpha$ -cuts of a given fuzzy set,  $A$ , is called a level set<sup>§</sup> of  $A$  by Klir & Yuan (1995), i.e.,

$$\Lambda_A = \{\alpha \mid A(x) = \alpha \text{ for some } x \in X\} \quad (2.5)$$

<sup>§</sup>In some publications e.g. (Yager 2008a) call the  $\alpha$ -cuts by the name level sets. In this thesis Klir & Yuan (1995) definition is adapted.



The first use of the mathematical definition of  $\alpha$ -cuts is rooted in Zadeh (1965), it was used to define the convexity of fuzzy sets. It is also used to define alternative definitions to some of the terms defined above.

**Definition 2.1.15** *Let  $A$  be a fuzzy set. Then,*

- *Convexity:  $A$  is convex if all its  $\alpha$ -cuts are convex.*
- *Support:  $\text{supp}(A) = A_{0+}$ .*
- *Core:  $\text{core}(A) = A_1$ .*
- *Normality:  $A$  is normal if  $A_1 \neq \emptyset$ .*

*Adapted from Klir & Yuan (1995)*

The support is a particular case where the strong  $\alpha$ -cut is used in the definition of the function rather than the  $\alpha$ -cut. The support is all the elements that have non-zero membership grades, or in other words all the elements with membership grades greater than zero, which is exactly the definition of the strong  $\alpha$ -cut at level  $\alpha = 0$ . In situations when the domain of a FS is continuous, and the FS is convex, then it is evident that the  $\alpha$ -cuts of the FS are also continuous and hence are intervals.

**Definition 2.1.16** *If a FS,  $A$ , is convex and continuous then  $A_\alpha \in \mathbf{I}(X)$  and  $A_\alpha = [\underline{x}_\alpha, \bar{x}_\alpha]$ .*

*Adapted from Klir & Yuan (1995)*

Another situation arises when the domain of the FS is continuous and the FS is non-convex, such situations result from performing some operations on FSs, a very popular operation is the union operation.

**Definition 2.1.17** *If a FS,  $A$ , is non-convex and yet continuous then  $A_\alpha$  is the union of distinct intervals, i.e.,*

$$A_\alpha = \bigcup_{i=1}^{n_\alpha} [\underline{x}_{i,\alpha}, \bar{x}_{i,\alpha}]$$

*where  $n_\alpha$  is the number of distinct intervals for each level  $\alpha$ .*

*Adapted from Yager & Filev (1999)*

If the domain of the fuzzy sets is discrete, then it is easy to construct its  $\alpha$ -cuts.

**Definition 2.1.18** *If a FS,  $A$ , is defined in a discrete and finite domain then  $A_\alpha$  is a discrete crisp, i.e.,*

$$A_\alpha = \{x_{1,\alpha}, x_{2,\alpha}, \dots, x_{n_\alpha,\alpha}\}$$

*where,  $n_\alpha$ , is the number of discrete elements at level  $\alpha$ .*

Adapted from Klir & Yuan (1995)

Figure 2.15 shows these situations of  $\alpha$ -cuts. Many properties have been defined for fuzzy sets using  $\alpha$ -cuts that are found useful to analyse and extend to further calculations.

**Theorem 2.1.3 ( $\alpha$ -cut properties)** *Let  $A, B \in F(X)$ . Then the following hold:*

1.  $A_{\alpha^+} \subseteq A_{\alpha}, \forall \alpha$
2.  $\alpha_1 \leq \alpha_2 \Rightarrow A_{\alpha_2} \subseteq A_{\alpha_1}$
3.  $A \subseteq B \Rightarrow A_{\alpha} \subseteq B_{\alpha}, \forall \alpha$
4.  $A = B \Rightarrow A_{\alpha} = B_{\alpha}, \forall \alpha$
5.  $(A \cup B)_{\alpha} = A_{\alpha} \cup B_{\alpha}$
6.  $(A \cap B)_{\alpha} = A_{\alpha} \cap B_{\alpha}$
7.  $A'_{\alpha} = (A_{(1-\alpha)^+})'$

Adapted from Klir & Yuan (1995)

**Proof.** The proof can be found in (Klir & Yuan 1995). ■

These properties are of great importance. The basic fuzzy set operations defined using the EP are shown to be equal to those defined using the  $\alpha$ -cut RT. This fact shows that the basic fuzzy set operations can be calculated using basic crisp set operations. The union and intersection are derived straight forward, i.e. the  $\alpha$ -cut of the union or intersection of two FSs is equal to the union of the  $\alpha$ -cuts of both sets. The complementation is another story, it is not straight forward and needs some manipulation. The  $\alpha$ -cut of the complement of a FS is equal to the complement of the strong  $\alpha$ -cut at level  $1 - \alpha$  of the FS. Klir & Yuan (1995) noticed that, " *This is not surprising since fuzzy sets violate, by definition, the two basic properties of the complement of crisp sets, the law of contradiction and the law of excluded middle*". The class of functions that can be derived directly following the union and intersection example are called cutworthy and will be demonstrated later in this section. The important part of this assertion is that these class of functions can be generalised using the extension principle. Zadeh defined this form of the EP using the  $\alpha$ -cut RT.

**Theorem 2.1.4 ( $\alpha$ -EP)** *The  $\alpha$ -cut version of the EP, ( $\alpha$ -EP), is defined as follows:*

$$B = f(A_1, \dots, A_n) = \bigcup_{\forall \alpha} \alpha f(A_{1,\alpha}, \dots, A_{n,\alpha}) \quad (2.6)$$

*Adapted from Zadeh (1975a)*

In general, Zadeh (1975a) was the first to point out that both forms of the EP are equal. Nguyen (1978) studied this claim and defined the necessary and sufficient conditions to satisfy such assertion. Later Araabi et al. (2001) discussed the relation between both EPs and showed that restrictions on the range of functions that can be extended by the  $\alpha$ -EP is more general than that of Zadeh's EP (i.e. sup-min composition).

**Corollary 2.1.2** *In general, theorems (2.1.1) and (2.1.4) are equal (Araabi et al. 2001, Nguyen 1978, Zadeh 1975a).*

Although this fact has been stated many times, Araabi et al. (2001) showed that only certain families of functions can satisfy this assertion. The family of functions that can be extended using  $\alpha$ -cuts satisfy the following property:

**Definition 2.1.19 (cutworthy functions)** *Let,  $A \in F(X)$ , be a FS and  $f$  be a monotone function such that*

$$(f(A))_{\alpha} = f(A_{\alpha})$$

*then  $f$  is called a cutworthy function.*

*Adapted from Araabi et al. (2001)*

In order for  $f$  to be cutworthy it has to satisfy the following restriction.

**Theorem 2.1.5** *Let,  $A$  and  $B \in C(X)$ , be two crisp sets then any function  $f$  such that*

$$A \subset B \Rightarrow f(A) \subset f(B)$$

*is a cutworthy function.*

*Adapted from Araabi et al. (2001)*

The range of functions that are defined as cutworthy and restricted by the crisp subsethood preservation are functions that result in a fuzzy set not a scalar value. Now, it is convenient to discuss the interpretation of fuzzy set membership grades with relation to  $\alpha$ -cuts. The fact that  $\alpha$ -cuts are crisp sets, allows them to be represented as restricted fuzzy sets. Let  $A \in F(X)$  be a FS, let also  $\lambda, \beta \in [0, 1]$  such that  $\lambda$  takes values that are always greater than any of the membership grades of  $A$ , i.e.,  $\lambda > A(x)$  at any  $x$ . On the other hand,  $\beta$  takes values that are always less than or equal to any of the membership grades of  $A$ , i.e.,  $\beta \leq A(x)$  at any  $x$ . Then any FS within the support of  $A$  with elements of the domain having membership grades that take values  $\beta$  are subsets of  $A$ . Also any FS outside the support of  $A$  with elements of the domain having membership grades that take

values  $\lambda$  are not subsets of  $A$ , but actually supersets of  $A$ . To explain this assertion, Figure 2.16 shows two FSs  $A', A'' \in \mathbf{F}(X)$ ,  $A'(x) = \beta, \forall x \in X$  and  $A''(x) = \lambda, \forall x \in X$ . Then it is clear that  $A' \subseteq A$  and  $A'' \supset A$ . The membership grade of an element of a FS is the maximum grade attained by that element for all its fuzzy subsets. The  $\alpha$ -cut representation is exactly the same, it is the union of all  $\alpha$ -FSs of  $A$ . The only difference is that  $\alpha$ -FSs are constrained fuzzy subsets of a FS. They are constrained by the order of level  $\alpha$ . It is easy to think of the fuzzy subsets of a particular FS as an embedded fuzzy set and that the  $\alpha$ -FSs are constrained embedded fuzzy sets. The main advantage of this constraining operation is that it allows the  $\alpha$ -FSs to have crisp set counterparts, i.e.,  $\alpha$ -cuts which makes it easy to extend operations from classical mathematics to fuzzy mathematics. This particular feature makes  $\alpha$ -cuts theoretically meaningful and practically tractable. The main focus of this thesis centered around the idea of  $\alpha$ -cuts and how to extend it to more complex extensions of FSs. Many extensions to FSs have been proposed in order to introduce more degrees of freedom and thus accommodate extra-levels of uncertainty. Interval valued fuzzy sets are amongst the most popular extensions of fuzzy sets. They are closely related to type-2 fuzzy sets and they are the subject of study in the following section.

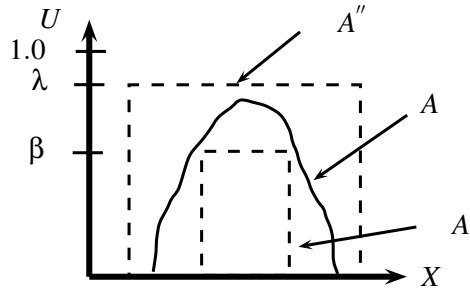


Fig. 2.16.  $A' \subseteq A$  and  $A'' \supset A$ .

## 2.2 Interval Valued Fuzzy Sets

Interval-valued fuzzy sets (IVFSs) are extensions of FSs. The development phases of the theory of IVFSs took different pathways (Mendel 2007), one of which preserved the name IVFS famously advocated by Gorzalczany (1987). Although traces of the theory dates back to 1975, some researchers, e.g. (Dubois & Prade 2005, Cornelis et al. 2006), point out that the concept of IVFSs have been introduced independently by different researchers in the same year. What is important to the context of this thesis is the close relationship between type-2 fuzzy sets and IVFSs. To this end Zadeh (1975a) suggested the concept of IVFSs while calculating the intersection of two type-2 fuzzy sets. The main difference between the concept of a FS and the concept of a IVFS is the modification to the grades of membership of the elements in the set. These grades are not

values in  $U$ , but are closed intervals in  $U$ .

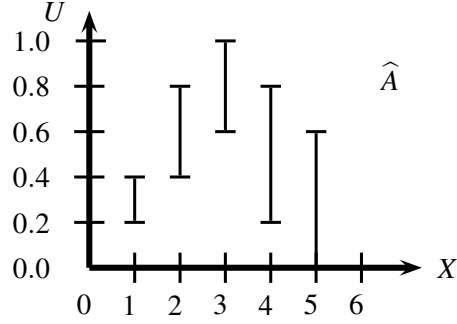


Fig. 2.17. Discrete IVFS  $\hat{A}$ .

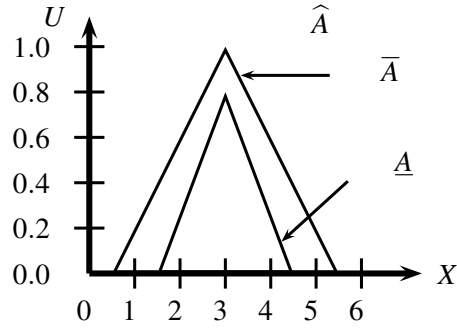


Fig. 2.18. The LMF  $\underline{A}$  and UMF  $\bar{A}$  of continuous IVFS  $\hat{A}$ .

**Definition 2.2.1 (IVFS)** An interval valued fuzzy set (IVFS),  $\hat{A}$ , over  $X$  is a function defined as follows:

$$\begin{aligned}\hat{A} : X &\rightarrow \mathbf{I}(U) \\ x &\mapsto [\underline{u}_x, \bar{u}_x]\end{aligned}$$

Let,  $\hat{F}(X)$ , be the set of all IVFSs on  $X$ . Here  $\hat{A}(x) = [\underline{u}_x, \bar{u}_x] \in \mathbf{I}(U)$  is an interval representing the grade of membership of  $x$  in  $\hat{A}$ . IVFSs can be described using the classical set representation as follows:

$$\begin{aligned}\hat{A} &= \left\{ (x, \hat{A}(x)) \mid x \in X, \hat{A}(x) \subseteq U \right\} \\ &= \left\{ (x, [\underline{u}_x, \bar{u}_x]) \mid x \in X, [\underline{u}_x, \bar{u}_x] \subseteq U \right\}\end{aligned}\tag{2.7}$$

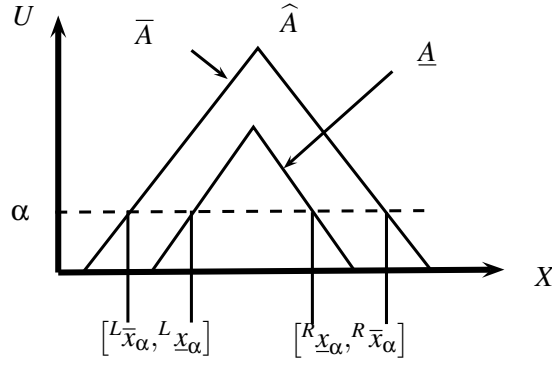


Fig. 2.19. The  $\alpha$ -cut of continuous IVFS  $\hat{A}$  using Kaufmann and Gupta method.

Alternatively, one can describe an IVFS using the union of the pairs containing each element  $x$  of the set and its associated interval membership grade.

$$\begin{aligned}\hat{A} &= \bigcup_{\forall x \in X} (x, \hat{A}(x)) , \hat{A}(x) \subseteq U \\ &= \bigcup_{\forall x \in X} (x, [\underline{u}_x, \bar{u}_x]) , [\underline{u}_x, \bar{u}_x] \subseteq U\end{aligned}\quad (2.8)$$

Due to the fact that all elements of the IVFS have interval membership grades, it is a common practice to define two fuzzy sets that represent the boundaries of the IVFS.

**Definition 2.2.2 (LMF)** A lower membership function (LMF) of an IVFS,  $\hat{A}$ , is a fuzzy set,  $\underline{A}$ , where  $\underline{A}(x) = \underline{u}_x, \forall x$ .

**Definition 2.2.3 (UMF)** An upper membership function (UMF) of an IVFS,  $\hat{A}$ , is a fuzzy set,  $\bar{A}$ , where  $\bar{A}(x) = \bar{u}_x, \forall x$ .

Hence, an IVFS can be represented using the LMF and UMF as follows:

**Definition 2.2.4** An IVFS  $\hat{A}$  is completely determined by its LMF and UMF, i.e.,

$$\hat{A} = (\underline{A}, \bar{A}) \quad (2.9)$$

which means  $\hat{A}(x) = [\underline{A}(x), \bar{A}(x)] , \forall x$ .

Once again IVFSs can be described using classical set representation as follows:

$$\begin{aligned}\hat{A} &= \{ (x, (\underline{A}, \bar{A})(x)) \mid x \in X , (\underline{A}, \bar{A})(x) \subseteq U \} \\ &= \bigcup_{\forall x \in X} (x, (\underline{A}, \bar{A})(x)) , (\underline{A}, \bar{A})(x) \subseteq U\end{aligned}\quad (2.10)$$

The fundamental set operations of fuzzy sets are extended to IVFSs by extending the single valued membership to intervals.

**Definition 2.2.5** *The following fundamental operations hold between IVFSs  $\hat{A}$  and  $\hat{B} \in \hat{F}(X)$  :*

$$(\hat{A} \cup \hat{B})(x) = [\underline{A}(x) \vee \underline{B}(x), \bar{A}(x) \vee \bar{B}(x)] \quad (\text{Union})$$

$$(\hat{A} \cap \hat{B})(x) = [\underline{A}(x) \wedge \underline{B}(x), \bar{A}(x) \wedge \bar{B}(x)] \quad (\text{Intersection})$$

$$\hat{A}'(x) = [1 - \bar{A}(x), 1 - \underline{A}(x)] \quad (\text{Complementation})$$

Zadeh (1975a) generalised the EP to operate with FSs with interval membership grades which can be interpreted as an EP for IVFSs.

**Definition 2.2.6 (IVEP)** *Let,  $X = X_1 \times \dots \times X_n$ , be the Cartesian product of universes, and  $\hat{A}_1, \dots, \hat{A}_n$  be IVFSs in each universe respectively where their respective LMFs are  $\underline{A}_1, \dots, \underline{A}_n$  and their respective UMFs are  $\bar{A}_1, \dots, \bar{A}_n$ . Also let  $Y$  be another universe and  $\hat{B} \in Y$  be an IVFS such that  $\hat{B} = f(\hat{A}_1, \dots, \hat{A}_n)$ , where  $f : X \rightarrow Y$  is a monotone mapping. Then the interval valued EP (IVEP) can be defined as follows:*

$$\hat{B} = (f(\underline{A}_1, \dots, \underline{A}_n), f(\bar{A}_1, \dots, \bar{A}_n))$$

This means, to derive operations for IVFSs one only needs to derive operations for their LMF and UMF.

### 2.2.1 Alpha-cuts of IVFSs

Zeng & Shi (2005), Zeng & Li (2006b) and Zeng et al. (2007) investigated the use of  $\alpha$ -cuts for IVFSs and how to extend arbitrary operations to such sets. They provided a host of definitions for  $\alpha$ -cuts for IVFSs and even FSs, all are summarised in Table 2.1.

Table 2.1. Definitions of IVFSs  $\alpha$ -cuts.

$\alpha$ -cut	Definition
$\hat{A}_{[\underline{\alpha}, \bar{\alpha}]}^{\geq, \geq}$	$\{x \mid \underline{A}(x) \geq \underline{\alpha}, \bar{A}(x) \geq \bar{\alpha}\}$
$\hat{A}_{[\underline{\alpha}, \bar{\alpha}]}^{\geq, >}$	$\{x \mid \underline{A}(x) \geq \underline{\alpha}, \bar{A}(x) > \bar{\alpha}\}$
$\hat{A}_{[\underline{\alpha}, \bar{\alpha}]}^{>, \geq}$	$\{x \mid \underline{A}(x) > \underline{\alpha}, \bar{A}(x) \geq \bar{\alpha}\}$
$\hat{A}_{[\underline{\alpha}, \bar{\alpha}]}^{>, >}$	$\{x \mid \underline{A}(x) > \underline{\alpha}, \bar{A}(x) > \bar{\alpha}\}$
$\hat{A}_{[\underline{\alpha}, \bar{\alpha}]}^{\leq, \leq}$	$\{x \mid \underline{A}(x) \leq \underline{\alpha}, \bar{A}(x) \leq \bar{\alpha}\}$
$\hat{A}_{[\underline{\alpha}, \bar{\alpha}]}^{\leq, <}$	$\{x \mid \underline{A}(x) \leq \underline{\alpha}, \bar{A}(x) < \bar{\alpha}\}$
$\hat{A}_{[\underline{\alpha}, \bar{\alpha}]}^{<, \leq}$	$\{x \mid \underline{A}(x) < \underline{\alpha}, \bar{A}(x) \leq \bar{\alpha}\}$
$\hat{A}_{[\underline{\alpha}, \bar{\alpha}]}^{<, <}$	$\{x \mid \underline{A}(x) < \underline{\alpha}, \bar{A}(x) < \bar{\alpha}\}$

In this table the first column represents the  $\alpha$ -cut notation of the IVFS,  $\hat{A} \in \hat{F}(X)$ , where the definition of the  $\alpha$ -cut is defined in the second column. For example, in the first case,  $\hat{A}_{[\underline{\alpha}, \bar{\alpha}]}^{\geq, \geq}$ , the subscript represent the level of the  $\alpha$ -cut i.e.,  $[\underline{\alpha}, \bar{\alpha}]$ , which is an interval. The use of an interval  $\alpha$ -cut rather than a single number is inspired by the interval membership grades of the IVFS. The superscript in the other hand defines the type of inequality used to restrict the domain values in relation to the LMF and UMF. A particular special case of these definitions is important. The one that uses the first case in Table 2.1 with a the condition that  $\alpha = \underline{\alpha} = \bar{\alpha}$ , i.e.,

$$\hat{A}_{\alpha} = \hat{A}_{[\underline{\alpha}, \bar{\alpha}]}^{\geq, \geq} = \{x \mid \underline{A}(x) \geq \alpha, \bar{A}(x) \geq \alpha\} \quad (2.11)$$

This forms a generalisation of the  $\alpha$ -cuts for FSs. Yager (2008b), also defined a closely related definition for IVFSs defined on discrete domains, which can easily be generalised for continuous domains. The following derivation of  $\alpha$ -cuts of IVFSs is adapted from Yager (2008b). For an IVFS,  $\hat{A} \in X$ , let,  $\Lambda_{\hat{A}}$ , be the set that contains all the distinct membership grades in set,  $\hat{A}$ .

$$\Lambda_{\hat{A}} = \left\{ \left[ \underline{\lambda}, \bar{\lambda} \right] \mid \hat{A}(x) = \left[ \underline{\lambda}, \bar{\lambda} \right] \text{ for some } x \right\} \quad (2.12)$$

Let,  $H$ , be a subset of  $\Lambda_{\hat{A}}$ , i.e.,  $H \subseteq \Lambda_{\hat{A}}$ . Now, a derived set that contains ordered intervals is defined as follows:

$$D = \bigcup_{\forall H/\subseteq \Lambda_{\hat{A}}} \min_{\forall [\underline{\lambda}, \bar{\lambda}] \in H} \left( \left[ \underline{\lambda}, \bar{\lambda} \right] \right) \quad (2.13)$$

Thus,  $D$ , contains all the distinct intervals,  $[\underline{\alpha}, \bar{\alpha}]$ , that are derived from set,  $\Lambda_{\hat{A}}$ . Then an  $\alpha$ -cut is a crisp set that contains all domain values with membership grades greater than or equal to the level  $[\underline{\alpha}, \bar{\alpha}]$ .

$$\hat{A}_{[\underline{\alpha}, \bar{\alpha}]} = \{x \mid \hat{A}(x) \geq [\underline{\alpha}, \bar{\alpha}], \forall x\} \quad (2.14)$$

Here  $\hat{A}_{[\underline{\alpha}, \bar{\alpha}]} \in \mathcal{C}(X)$  is a crisp set. Then, the IVFS  $\hat{A}$  can be represented as follows:

$$\hat{A} = \bigcup_{\forall [\underline{\alpha}, \bar{\alpha}] \in D} [\underline{\alpha}, \bar{\alpha}] \hat{A}_{[\underline{\alpha}, \bar{\alpha}]} \quad (2.15)$$

where  $[\underline{\alpha}, \bar{\alpha}] \hat{A}_{[\underline{\alpha}, \bar{\alpha}]}$ , is an IVFS with membership grade  $[\underline{\alpha}, \bar{\alpha}] \wedge \hat{A}_{[\underline{\alpha}, \bar{\alpha}]}(x)$  and  $\hat{A}_{[\underline{\alpha}, \bar{\alpha}]}(x) = 1$  if  $x \in \hat{A}_{[\underline{\alpha}, \bar{\alpha}]}$  or zero otherwise. This representation is also valid for the  $\alpha$ -cut definitions in Table 2.1 by replacing  $\hat{A}_{[\underline{\alpha}, \bar{\alpha}]}$  with any of the definitions provided in the first column. where  $\forall \alpha : {}^L \bar{x}_{\alpha} \leq {}^L x_{\alpha} \leq {}^R x_{\alpha} \leq {}^R \bar{x}_{\alpha}$ , and  $h(\underline{A}) = \sup_{\forall x} \underline{A}(x)$  is the height of LMF. Another method of defining  $\alpha$ -cuts for IVFSs is the method provided by Kaufmann & Gupta (1985). It is also used by Wu & Mendel (2007a) and Wu & Mendel (2008a) to define the aggregation operation for IVFSs<sup>¶</sup>. To explain this

<sup>¶</sup>IVFSs in those papers are called interval type-2 fuzzy sets.



method assume that  $\hat{A}$  is a continuous and convex IVFS i.e.  $\underline{A}$  and  $\bar{A}$  are continuous and convex. Furthermore, let the  $\alpha$ -cut of the LMF be  $\underline{A}_\alpha = [{}^L\underline{x}_\alpha, {}^R\underline{x}_\alpha]$ , and of the UMF be  $\bar{A}_\alpha = [{}^L\bar{x}_\alpha, {}^R\bar{x}_\alpha]$ . Then,  $\hat{A}_\alpha$ , is shown in Figure 2.19 and described in the following manner, i.e.,

$$\hat{A}_\alpha^{KG} = \begin{cases} [{}^L\bar{x}_\alpha, {}^L\underline{x}_\alpha], [{}^R\underline{x}_\alpha, {}^R\bar{x}_\alpha] & , \alpha < h(\underline{A}) \\ [{}^L\bar{x}_\alpha, {}^R\bar{x}_\alpha] & , \alpha \geq h(\underline{A}) \end{cases} \quad (2.16)$$

In this thesis a different approach is developed to define  $\alpha$ -cuts for IVFSs. This is represented in Section 3.1.

### 2.2.2 Relation to Atanassov's Intuitionistic Fuzzy Sets

Many generalisations have been developed to extend fuzzy sets, and IVFSs are one of the famous and well established generalisations of fuzzy sets (Klir & Yuan 1995). Other terms have been developed such as Atanassov's Intuitionistic Fuzzy Sets (**A-IFSs**) (Atanassov 1986), Grey Sets, Vague Sets and Clouds. These sets all turned out to be representable by IVFSs (Bustince & Burillo 1996, Deschrijver & Kerre 2003, Dubois & Prade 2005). Also there is a strong relation between IVFSs, Fuzzy Rough Sets, Rough Fuzzy Sets which is interesting and left for further studies. Within this thesis only A-IFSs are considered, because some important definitions and operations, such as the measures of uncertainty in Chapter 4 of this thesis, are applied to A-IFSs and can be equally used for IVFSs.

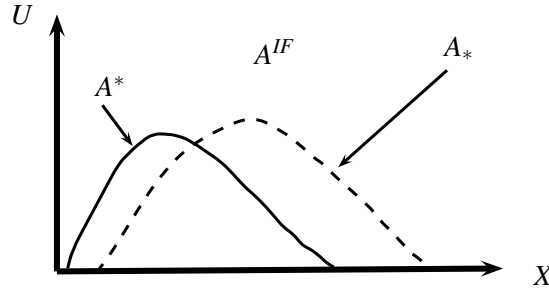


Fig. 2.20. Intuitionistic FS  $A^{IF}$  with membership function  $A_*$ , and nonmembership function  $A^*$ .

**Definition 2.2.7 (A-IFS)** *Atanassov's intuitionistic fuzzy sets are sets that are characterised by two membership functions, one represents the degree of membership and the other represents the degree of non-membership. This is defined as follows:*

$$A^{IF} = \{(x, A_*(x), A^*(x)) \mid \forall x \in X; A_*(x), A^*(x) \in [0, 1]\}$$

where  $0 \leq A_*(x) + A^*(x) \leq 1 \mid \forall x \in X$ ,  $A_*(x)$  is the degree of membership, and  $A^*(x)$  is the degree of non-membership.

*Adapted from Atanassov (1986)*

Figure 2.20 shows an IFS, it is clear that A-IFSs are another generalisation of FSs viewed as  $A \equiv A^{IF}$  if  $A_*(x) = 1 - A^*(x)$ . An important term in A-IFSs is the *Intuitionistic Index* defined by  $\pi_{A^{IF}}(x) = 1 - A_*(x) - A^*(x)$  which is zero in case of FSs. An important result presented in (Atanassov & Gargov 1989) is the development of a mapping function that enables describing any A-IFS in terms of an IVFS and vice versa. Also in a more concise manner (Deschrijver & Kerre 2003) proved that by definition A-IFSs are equal to IVFSs<sup>||</sup>. These facts support the use of definitions and operations that are basically defined for A-IFSs with IVFSs. However, the real importance of such definition is that it provides another way of constructing IVFSs from a totally different perspective, i.e., gives alternative semantics from the same mathematical formulation, e.g., constructing the A-IFS from positive data (degree of membership) and negative data (degree of non-membership).

## 2.3 Type-2 Fuzzy Sets

Zadeh (1975a) defined type-2 fuzzy sets (**T2FS**) to be sets with elements that have memberships that themselves are fuzzy sets. Zadeh explained that this move was “*motivated by the close association which exists between the concept of a linguistic truth with truth-values such as true, quite true, very true, more or less true, etc., on the one hand, and fuzzy sets in which the grades of membership are specified in linguistic terms such as low, medium, high, very low, not low and not high, etc., on the other*”. Some researchers (e.g. Chen & Kawase (2000)) call such sets, fuzzy-valued fuzzy sets (or fuzzy-fuzzy sets) referring to their membership function being fuzzy values (i.e. sets). In this thesis the following functional definition of T2FSs is adapted.

**Definition 2.3.1 (T2FS)** A Type-2 fuzzy set (**T2FS**),  $\tilde{A}$ , over  $X$  is defined by the following function:

$$\tilde{A} : X \rightarrow \mathbf{F}(U) \tag{2.17}$$

where  $U = [0, 1]$  is the domain of membership.

*Adapted from Aisbett et al. (2010)*

Let,  $\tilde{F}(X)$ , be the set of all T2FSs defined on universe  $X$ . It is clear that the the grade of membership of any element  $x \in X$  is a FS  $\tilde{A}(x) \in \mathbf{F}(U)$ . The following definitions and discussion are adapted

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<sup>||</sup>It is beyond the scope of this discussion to recall the proof.

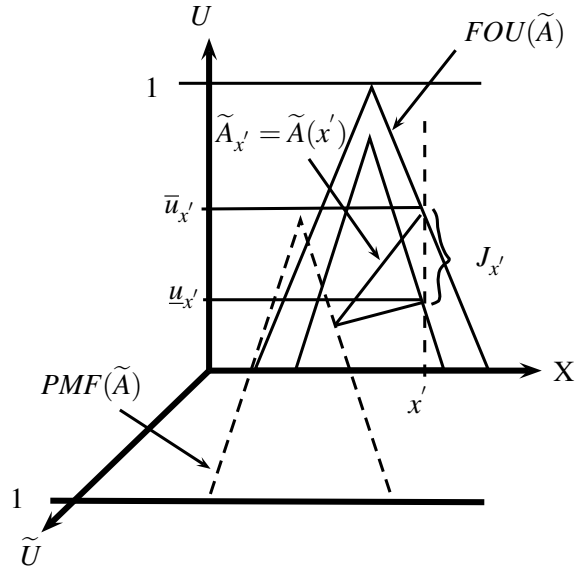


Fig. 2.21. 3D representation of a T2FS with triangular vertical slices.

from Mendel (2001) and Mendel & John (2002), but using the standard mathematical notation following the IVFS and FS notations of earlier sections\*\*. Using the standard set theory union a T2FS can be represented by the following:

$$\tilde{A} = \bigcup_{\forall x \in X} (x, \tilde{A}(x)) \quad (2.18)$$

Because the membership of the T2FS is a FS it consists of membership domain values and corresponding membership grades. Each membership domain value  $u_x$  is an element of the universe  $U$  and called a *primary grade (PG)*. Each primary grade is assigned a membership grade  $\tilde{u}_x$  called a *secondary grade (SG)* and defined on another universe  $\tilde{U} = [0, 1]$ . Although both  $U$  and  $\tilde{U}$  are equal to the unit interval, they are distinguished from each other to reflect the fact that they are in different dimensions. The same justification applies to the PGs  $u_x$  and SGs  $\tilde{u}_x$ . Figure 2.21 shows a 3D depiction of a T2FS with triangular vertical slices, and Figure 2.22 is a 2D representation of the same T2FS. For clarity Figure 2.23 shows a discrete T2FS with primary and secondary grades. The starting point of any T2FS definition is the following definition:

$$\tilde{A} = \left\{ ((x, u_x), \tilde{u}_x) \mid \forall x \in X, u_x \in U, \tilde{u}_x \in \tilde{U} \right\} \quad (2.19)$$

\*\*This move to standard mathematical notations already advocated by Walker & Walker (2005) and Aisbett et al. (2010). The discussion at the end of this chapter elaborates on this issue.

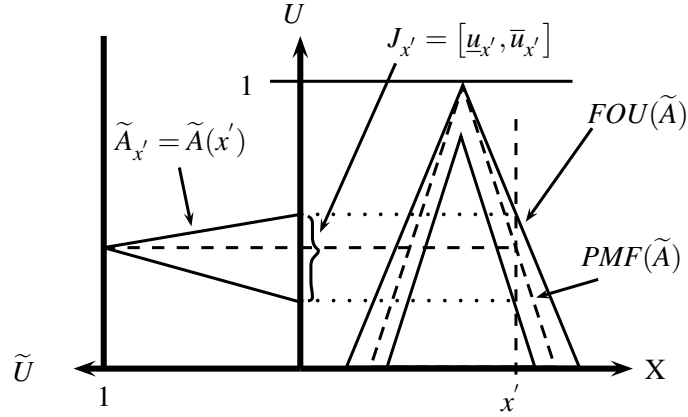


Fig. 2.22. 2D representation of the T2FS with triangular vertical slices of Figure 2.21.

The function that represents the T2FS membership grade at domain value  $x \in X$  is denoted by  $\tilde{A}_x$  rather than  $\tilde{A}(x)$  for simplicity and defined as follows:

$$\begin{aligned} \tilde{A}_x : U &\rightarrow \tilde{U} \\ u_x &\mapsto \tilde{u}_x \end{aligned} \quad (2.20)$$

Clearly,  $\tilde{u}_x = \tilde{A}_x(u_x) \in \tilde{U}$  represents the SG of PG  $u_x$ . Because  $\tilde{A}_x$  is a FS it can equally be represented in the same manner as the FS in Eq. 2.1.

$$\tilde{A}_x = \bigcup_{\forall u_x \in U} (u_x, \tilde{u}_x) \quad (2.21)$$

Substituting  $\tilde{A}(x)$  with  $\tilde{A}_x$  in Eq. 2.18 will produce a relatively simple definition.

$$\tilde{A} = \bigcup_{\forall x \in X} (x, \tilde{A}_x) \quad (2.22)$$

consequently, substituting the value of  $\tilde{A}_x$  from Eq. 2.21 in Eq. 2.22 will produce a more decomposed form.

$$\tilde{A} = \bigcup_{\forall x \in X} \left( x, \bigcup_{\forall u_x} (u_x, \tilde{u}_x) \right) \quad (2.23)$$

In Mendel & John (2002),  $\tilde{A}_x$ , is called the *vertical slice* (**VS**) because of its 2D nature and is formally defined as follows:

**Definition 2.3.2 (Vertical Slice)** A vertical slice of a T2FS,  $\tilde{A}$ , is the T2FS membership at a single

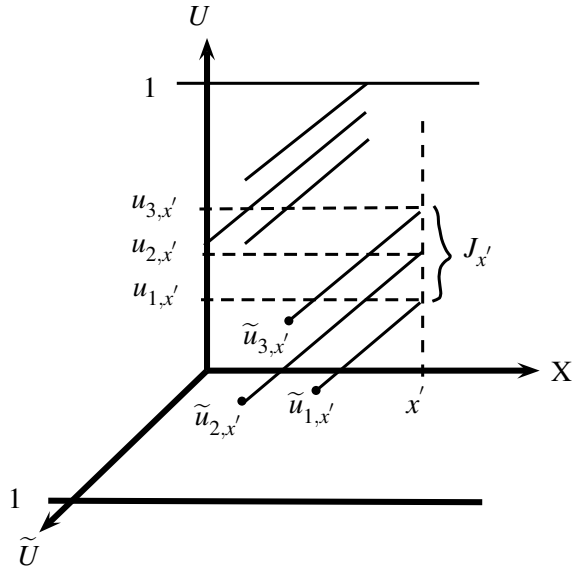


Fig. 2.23. 3D representation of a discrete T2FS with discrete vertical slices.

value  $x$ . The VS is a FS,  $\tilde{A}_x \in \mathbf{F}(U)$ .

*Adapted from Mendel & John (2002)*

Eq. 2.23 is actually the VS representation Theorem (**RT**) defined in Mendel & John (2002).

**Theorem 2.3.1 (VS RT)** *The vertical slice representation theorem (VS RT) of a T2FS,  $\tilde{A} \in \tilde{\mathbf{F}}(X)$ , is the union of all its vertical slices, i.e.,*

$$\tilde{A} = \bigcup_{\forall x \in X} (x, \tilde{A}_x) \quad (2.24)$$

*Adapted from Mendel & John (2002)*

Eq. 2.23 represents a decomposition of the VS RT. Some standard terms for T2FSs were defined by Mendel & John (2002) that are found useful to communicate the complex nature of these sets. Some of these terms are redefined in light of the VS RT.

**Definition 2.3.3 (Primary Grade)** *A PG  $u_x$  in a T2FS,  $\tilde{A} \in \tilde{\mathbf{F}}(X)$ , is a single domain value of the VS,  $\tilde{A}_x$ , at element  $x \in X$ .*

*Adapted from Mendel & John (2002)*

**Definition 2.3.4 (Secondary Grade)** *A SG  $\tilde{u}_x$  in a T2FS,  $\tilde{A} \in \tilde{\mathbf{F}}(X)$ , is the membership grade of a single PG  $u_x$  of VS  $\tilde{A}_x$ , i.e.,  $\tilde{u}_x = \tilde{A}_x(u_x)$ .*

Adapted from Mendel & John (2002)

**Definition 2.3.5 (Primary Membership)** The primary membership (**PM**)  $J_x$  in a T2FS,  $\tilde{A} \in \tilde{F}(X)$ , is the support of VS  $\tilde{A}_x$ , i.e.,

$$J_x = \text{supp}(\tilde{A}_x) \quad (2.25)$$

Adapted from Aisbett et al. (2010)

In discrete T2FSs, the PM consists of exactly  $q$  PGs, i.e.,  $J_x = \{u_{1,x}, u_{2,x}, \dots, u_{q,x}\}$ . In continuous spaces the PM is bound between two PGs, i.e.,  $J_x = [\underline{u}_x, \bar{u}_x]$ .

**Definition 2.3.6 (Footprint Of Uncertainty)** The footprint of uncertainty (**FOU**) of a T2FS,  $\tilde{A} \in \tilde{F}(X)$ , is the union of all the PMs of that set, i.e.,

$$FOU(\tilde{A}) = \bigcup_{\forall x} (x, J_x) \quad (2.26)$$

Adapted from Mendel & John (2002)

Note that normally the FOU is considered to be IVFSs, meaning that  $J_x$  is an interval (Mendel & John 2002). In discrete spaces the FOU is constructed by redefining the PMs by taking the infimum and supremum of the discrete PMs. This can be demonstrated by the following equation:

$$J_x = \left[ \inf_{\forall i} u_{i,x}, \sup_{\forall i} u_{i,x} \right], \quad i = 1, 2, \dots, q \quad (2.27)$$

This fact is implied from Mendel & John (2002), where in the definition of the FOU the primary membership is assumed to be a bounded subset of  $U$ . What is interesting at this point is the relationship between T2FSs and IVFSs, in which, IVFSs are special cases of T2FSs. To discuss this fact, first the formal definition of IT2FSs is defined next.

**Definition 2.3.7 (Interval T2FS)** An interval type-2 fuzzy set (**IT2FS**) is defined to be a T2FS where all the secondary grades are at unity, i.e.,  $\forall \tilde{u}_x = 1$ ,  $\forall u_x \in J_x$ ,  $\forall x \in X$ .

Adapted from Mendel & John (2002)

Two important facts result from this definition.

**Corollary 2.3.1** The following is true:

- An IT2FS can be completely determined using its FOU (Mendel et al. 2006).
- An IT2FS is the same as an IVFS (Bustince et al. 2009, Mendel 2007).

The first fact is discussed clearly by Mendel et al. (2006), and the second is a direct consequence from the fact that the *FOU* is actually an IVFS. In practice <sup>††</sup>, IT2FSs are treated as IVFSs. Another term that is not used commonly in the literature, but of particular interest to this thesis is the *principal membership function*. Karnik & Mendel (2001) first introduced the term as, “*the set of primary memberships having secondary membership equal to 1*”. In a later definition by Mendel (2001), a more restrictive form of the term is defined.

**Definition 2.3.8 (PMF)** Assume that each of the secondary membership functions (i.e. VSs) of a type-2 fuzzy set has only one secondary grade that equals 1. A principal membership function (PMF) is the union of all such points at which this occurs, i.e., the PMF is a fuzzy set defined as follows:

$$PMF(\tilde{A}) = \{(x, u_x) \mid x \in X, \tilde{u}_x = 1 \text{ for only one } u_x \in J_x\} \quad (2.28)$$

*Adapted from Mendel (2001)*

In this view the PMF is advocated to reflect the certainty of a T2FS, “*when all membership function uncertainties disappear, a type-2 membership function reduces to a principal membership function*”(Mendel 2001). For interval secondary membership functions, the PMF is formed by the union of all the PM mid-points. In this thesis, the generalised view first stated by Karnik & Mendel (2001) is found to be useful, and hence a new term is introduced below.

**Definition 2.3.9 (Principal Set)** A principal set (PS) is the set (either fuzzy set or IVFS) of primary grades having secondary grades equal to 1, i.e.,

$$PS(\tilde{A}) = \{(x, u_x) \mid x \in X, \tilde{u}_x = 1, u_x \in J_x\} \quad (2.29)$$

The PMF is a special case of the PS. It is the condition when there is only one PG having a SG equals to unity. The PS can be a fuzzy set, or even an IVFS. It still represents the certainty of the T2FS, and help elevate the relationship between a T2FS and special cases, i.e., IVFSs, FSs, intervals and crisp sets.

## 2.4 Operations using Vertical Slices

The EP has been used by Zadeh (1975a) and Mizumoto & Tanaka (1976) to derive the intersection and union of T2FSs. Karnik & Mendel (2001) provide an in-depth investigation on these operations.

**Theorem 2.4.1 (T2 EP)** Let,  $X = X_1 \times \dots \times X_n$ , be the Cartesian product of universes, and  $\tilde{A}_1, \dots, \tilde{A}_n$  be T2FSs in each respective universe. Also let  $Y$  be another universe and  $\tilde{B} \in Y$  be a T2FS

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<sup>††</sup>e.g. see (Mendel 2007, Mendel 2001, John & Coupland 2007) for a survey.

such that  $\tilde{B} = f(\tilde{A}_1, \dots, \tilde{A}_n)$ , where  $f : X \rightarrow Y$  is a monotone mapping. Then applying the EP to T2FSs (**T2EP**) leads to the following:

$$\begin{aligned}\tilde{B} \Leftrightarrow \tilde{B}(y) &= \sup_{(x_1, \dots, x_n) \in f^{-1}(y)} \min(\tilde{A}_1(x_1), \dots, \tilde{A}_n(x_n)) \\ &= \sup_{(x_1, \dots, x_n) \in f^{-1}(y)} \min(\tilde{A}_{1,x_1}, \dots, \tilde{A}_{n,x_n})\end{aligned}$$

where  $y = f(x_1, \dots, x_n)$ .

*Adapted from Mendel & John (2002)*

Although it may seem straight forward, the complexity of this definition is inherent in the determination the VS of the solution. The resulting VS which is a FS is evaluated using the t-norm operation across all possible VS combinations. To explain, using the minimum operation is computed between fuzzy sets rather than real values or intervals, which is it self another sup-min composition. Assume  $A = \bigcup_{a \in X} (a, A(a))$  and  $B = \bigcup_{b \in X} (b, B(b))$  are two FSs, then the minimum of these sets, i.e.  $C = \min(A, B)$  can be calculated using the EP as follows:

$$C(c) = \sup_{c=a \wedge b} A(a) \wedge B(b) \quad (2.30)$$

A comprehensive discussion can be found in (Mizumoto & Tanaka 1976, John 1998, Karnik & Mendel 2001, Mendel & John 2002). Let,  $\tilde{A}$ , be a discrete  $\ddagger\ddagger$  T2FS defined as follows:

$$\tilde{A} = \bigcup_{i_a=1}^{N_a} \left( x_{i_a}, \bigcup_{k_a=1}^{M_{i_a}} (u_{x_{i_a}}^{k_a}, \tilde{u}_{x_{i_a}}^{k_a}) \right) \quad (2.31)$$

where  $N_a$  is the number of discrete domain values  $x_{i_a}$  ;  $i_a = 1, 2, \dots, N_a$ ,  $M_{i_a}$  is the number of discrete PMs  $J_{x_{i_a}}$  for each domain value  $x_{i_a}$  consisting of  $u_{x_{i_a}}^{k_a}$  which represent the  $k_a^{th}$  PG of the  $i_a^{th}$  domain value ;  $k_a = 1, 2, \dots, M_{i_a}$ . Each SG  $\tilde{u}_{x_{i_a}}^{k_a} = \tilde{A}_{x_{i_a}}(u_{x_{i_a}}^{k_a})$  is associated with only one PG, meaning that there are  $M_{i_a}$  SGs for each domain value. The same interpretation is applied to another T2FS. Let,  $\tilde{B}$ , be another discrete T2FS defined as follows:

$$\tilde{B} = \bigcup_{i_b=1}^{N_b} \left( x_{i_b}, \bigcup_{k_b=1}^{M_{i_b}} (u_{x_{i_b}}^{k_b}, \tilde{u}_{x_{i_b}}^{k_b}) \right) \quad (2.32)$$

where  $N_b$  is the number of discrete domain values  $x_{i_b}$  ;  $i_b = 1, 2, \dots, N_b$ ,  $M_{i_b}$  is the number of discrete PMs  $J_{x_{i_b}}$  for each domain value  $x_{i_b}$ ,  $u_{x_{i_b}}^{k_b}$  represent the  $k_b^{th}$  PG of the  $i_b^{th}$  domain value ;

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$\ddagger\ddagger$  Here discrete sets are used for clarity purpose as it gives insight to the amount of computation required. Additionally, they are the cases actually used in applications.



$k_b = 1, 2, \dots, M_{i_b}$ , and  $\tilde{u}_{x_{i_b}}^{k_b}$  are the SGs. A VS of  $\tilde{A}$  at point  $x_{i_a}$  is  $\tilde{A}_{x_{i_a}}$  and a VS of  $\tilde{B}$  at point  $x_{i_b}$  is  $\tilde{B}_{x_{i_b}}$ . Then each VS is defined as follows:

$$\tilde{A}_{x_{i_a}} = \bigcup_{k_a=1}^{M_{i_a}} \left( u_{x_{i_a}}^{k_a}, \tilde{u}_{x_{i_a}}^{k_a} \right)$$

$$\tilde{B}_{x_{i_b}} = \bigcup_{k_b=1}^{M_{i_b}} \left( u_{x_{i_b}}^{k_b}, \tilde{u}_{x_{i_b}}^{k_b} \right)$$

The following basic set operations are defined.

**Definition 2.4.1 (Union)**

$$\begin{aligned} \tilde{A} \cup \tilde{B} &= \bigcup_{\forall x} \left( x, \left( \tilde{A} \cup \tilde{B} \right) (x) \right) \\ &= \bigcup_{\forall x} \left( x, \bigcup_{k_a=1}^{M_{i_a}} \bigcup_{k_b=1}^{M_{i_b}} \left( (u_x^{k_a} \vee u_x^{k_b}), (\tilde{u}_x^{k_a} \wedge \tilde{u}_x^{k_b}) \right) \right) \end{aligned} \quad (2.33)$$

where  $\bigcup_{k_a=1}^{M_{i_a}} \bigcup_{k_b=1}^{M_{i_b}}$  means the union over all admissible PGs  $u_x^{k_a}$  and  $u_x^{k_b}$ .

Mizumoto & Tanaka (1976) called the operation between the T2FS membership grades  $(\tilde{A} \cup \tilde{B})(x)$  the *Join* and denoted  $\tilde{A}(x) \sqcup \tilde{B}(x)$ , and since these membership grades are the VSs of the T2FS they can alternatively be denoted  $\tilde{A}_x \sqcup \tilde{B}_x$ .

**Definition 2.4.2 (Intersection)**

$$\begin{aligned} \tilde{A} \cap \tilde{B} &= \bigcup_{\forall x} \left( x, \left( \tilde{A} \cap \tilde{B} \right) (x) \right) \\ &= \bigcup_{\forall x} \left( x, \bigcup_{k_a=1}^{M_{i_a}} \bigcup_{k_b=1}^{M_{i_b}} \left( (u_x^{k_a} \wedge u_x^{k_b}), (\tilde{u}_x^{k_a} \wedge \tilde{u}_x^{k_b}) \right) \right) \end{aligned} \quad (2.34)$$

Mizumoto & Tanaka (1976) called the operation between the T2FS membership grades  $(\tilde{A} \cap \tilde{B})(x)$  the *Meet* and denoted  $\tilde{A}(x) \sqcap \tilde{B}(x)$ , and since these membership grades are the VSs of the T2FS they can alternatively be denoted  $\tilde{A}_x \sqcap \tilde{B}_x$ .

**Definition 2.4.3 (Complement)**

$$\begin{aligned} (\tilde{A})' &= \bigcup_{\forall x} \left( x, (\tilde{A})' (x) \right) \\ &= \bigcup_{\forall x} \left( x, \bigcup_{k_a=1}^{M_{i_a}} \left( (1 - u_x^{k_a}), \tilde{u}_x^{k_a} \right) \right) \end{aligned} \quad (2.35)$$

Mizumoto & Tanaka (1976) called the operation  $(\tilde{A})'(x)$  the *Negation* and denoted  $\neg\tilde{A}(x)$  or alternatively  $\neg\tilde{A}_x$ . A natural extension of these definitions to the special case of T2FSs, i.e. IT2FSs, can be directly achieved through the fact that all the secondary grades of the sets involved in the computation are at unity.

**Definition 2.4.4**

$$\begin{aligned}\tilde{A} \cup \tilde{B} &= \bigcup_{\forall x} \left( x, \bigcup_{k_a=1}^{M_{ia}} \bigcup_{k_b=1}^{M_{ib}} ((u_x^{k_a} \vee u_x^{k_b}), 1) \right) \\ \tilde{A} \cap \tilde{B} &= \bigcup_{\forall x} \left( x, \bigcup_{k_a=1}^{M_{ia}} \bigcup_{k_b=1}^{M_{ib}} ((u_x^{k_a} \wedge u_x^{k_b}), 1) \right) \\ (\tilde{A})' &= \bigcup_{\forall x} \left( x, \bigcup_{k_a=1}^{M_{ia}} ((1 - u_x^{k_a}), 1) \right)\end{aligned}\tag{2.36}$$

where  $\bigcup_{k_a=1}^{M_{ia}} \bigcup_{k_b=1}^{M_{ib}}$  means the union over all admissible PGs  $u_x^{k_a}$  and  $u_x^{k_b}$ .

## 2.5 The Wavy Slice Representation

Mendel & John (2002) defined operations for T2FSs without the use of the EP. Their method relies on decomposing a T2FS to a collection of simpler embedded type-2 fuzzy sets. Each embedded type-2 fuzzy set has only one embedded type-1 fuzzy set or simply embedded fuzzy set. The motivation behind this work was to avoid calculating operations using the complicated EP hoping to make them easy to formulate using already known methods from classical FSs. However, the amount of computation required by this representation is astronomically high. To explain this representation theorem, first the formal definitions of the embedded fuzzy sets and embedded type-2 fuzzy sets are reviewed. In order to be consistent with the material presented in the rest of this thesis, a functional interpretation of the original definition is provided.

**Definition 2.5.1 (EFS)** An embedded fuzzy set (EFS),  $A_e$ , of a T2FS  $\tilde{A} \in \tilde{F}(X)$  is defined as follows:

$$\begin{aligned}A_e : X &\rightarrow U \\ x &\mapsto u_x \quad | \quad \text{for only one } u_x \in J_x\end{aligned}\tag{2.37}$$

For demonstration, let  $\tilde{A}$ , be a discrete T2FS. An EFS,  $A_e$ , in  $\tilde{A}$  can be seen as follows:

$$A_e = \bigcup_{i=1}^N (x_i, u_{x_i})\tag{2.38}$$

where  $A_e$  has  $N$  domain values associated with exactly one grade from each PM, i.e.,  $J_1, J_2, \dots, J_N$ , namely  $u_{x_1}, u_{x_2}, \dots, u_{x_N}$ , and there are a total of  $n = \prod_{i=1}^N M_i$  of  $A_e$  embedded in  $\tilde{A}$  where  $M_i$  is the number of PGs associated with domain value  $x_i$ . Note that in continuous spaces there are uncountable number of embedded sets.

**Definition 2.5.2 (Wavy Slice)** *An embedded type-2 fuzzy set  $\tilde{A}_e$  also called the wavy slice (WS) is defined as follows:*

$$\begin{aligned} \tilde{A}_e : X \rightarrow U \rightarrow \tilde{U} : \\ : x \mapsto u_x \mapsto \tilde{u}_x \quad | \quad \text{for only one } u_x \in J_x \end{aligned} \quad (2.39)$$

A WS,  $\tilde{A}_e$ , in a discrete T2FS,  $\tilde{A}$ , can be seen as follows:

$$\tilde{A}_e = \bigcup_{i=1}^N (x_i, (u_{x_i}, \tilde{u}_{x_i})) \quad (2.40)$$

where  $\tilde{A}_e$  has  $N$  domain values associated with exactly one grade from each PM, i.e.,  $J_1, J_2, \dots, J_N$ , namely  $u_{x_1}, u_{x_2}, \dots, u_{x_N}$ , and there are a total of  $n = \prod_{i=1}^N M_i$  of  $\tilde{A}_e$  embedded in  $\tilde{A}$  where  $M_i$  is the number of PGs associated with domain value  $x_i$ . The formal definition of the WS representation theorem can be defined as follows:

**Theorem 2.5.1 (WSRT)** *The wavy slice representation theorem (WSRT) of a T2FS,  $\tilde{A}$ , is the union of all the wavy slices embedded in that set, i.e.,*

$$\tilde{A} = \bigcup_{\forall j} \tilde{A}_e^j \quad (2.41)$$

*Adapted from Mendel & John (2002)*

The discrete T2FS,  $\tilde{A} \in \tilde{F}(X)$ , can be represented as follows:

$$\tilde{A} = \bigcup_{j=1}^n \bigcup_{i=1}^N (x_i, (u_{x_i}, \tilde{u}_{x_i})) \quad (2.42)$$

### 2.5.1 Operations using wavy slices

In fact this is the easiest way to theoretically perform operations between T2FSs. Let,  $\tilde{A}$ , be a T2FS defined as follows:

$$\tilde{A} = \bigcup_{j_a=1}^{n_a} \tilde{A}_e^{j_a} \quad (2.43)$$

and,  $\tilde{B}$ , be another T2FS defined as follows:

$$\tilde{B} = \bigcup_{j_b=1}^{n_b} \tilde{B}_e^{j_b} \quad (2.44)$$

then the following operations can be defined.

**Definition 2.5.3**

$$\begin{aligned} \tilde{A} \cup \tilde{B} &= \bigcup_{j_a=1}^{n_a} \bigcup_{j_b=1}^{n_b} \tilde{A}_e^{j_a} \cup \tilde{B}_e^{j_b} \\ \tilde{A} \cap \tilde{B} &= \bigcup_{j_a=1}^{n_a} \bigcup_{j_b=1}^{n_b} \tilde{A}_e^{j_a} \cap \tilde{B}_e^{j_b} \\ (\tilde{A})' &= \bigcup_{j_a=1}^{n_a} (\tilde{A}_e^{j_a})' \end{aligned} \quad (2.45)$$

Although the representation is direct the computation is very complex.

## 2.6 The Geometric Representation

Modeling T2FSs in terms of geometric primitives was introduced by Coupland & John (2007) through the use of algorithms from computational geometry. The main idea is, “to represent a TIFS  $A$  over a continuous domain  $X$  as a set of connected straight line segments that need not be equally spaced across the domain”, as stated in Coupland & John (2007).

**Definition 2.6.1 (Geometric T1FS)** A geometric type-1 fuzzy set (**GT1FS**),  $A$ , is a 2D geometric object (i.e. polygon) on the  $X$  and  $U$  axes that is constructed of a poly-line of vertices made up by Piecewise Linear Functions (PLF). Accordingly the membership function of the FS,  $A$ , is defined as follows:

$$A(x) = \begin{cases} 0 & x < x_1 \text{ OR } x > x_n \\ y_k & x = x_k \\ y_k + \frac{x-x_k}{x_{k+1}-x_k}(y_{k+1}-y_k) & x_k < x < x_{k+1} \end{cases} \quad (2.46)$$

*Adapted from Coupland & John (2007)*

For T2FSs the main idea is to treat the vertical slices of a partially discrete T2FS in the same manner to the GT1FS case. In the other hand IVFSs are described by two GT1FSs representing the LMF and UMF. Coupland & John (2008a) provide a clear definition of a geometric type-2 fuzzy set (GT2FS).

**Definition 2.6.2** A geometric type-2 fuzzy set (**GT2FS**) is a collection of  $n$  triangles in 3D space forming a polyhedron, i.e.,

$$\tilde{A} = \bigcup_{i=1 \dots n} t^i, t^i = \begin{bmatrix} x_1^i & u_{x_1^i} & \tilde{A}_{x_1^i}(u_{x_1^i}) \\ x_2^i & u_{x_2^i} & \tilde{A}_{x_2^i}(u_{x_2^i}) \\ x_3^i & u_{x_3^i} & \tilde{A}_{x_3^i}(u_{x_3^i}) \end{bmatrix} \quad (2.47)$$

*Adapted from Coupland & John (2008a)*

the preference of a polyhedron is for their most simple design and processing. Other shapes may be used rather than triangles, although their planar nature makes them advantageous, for an IT2FS case it is clear that the third column is set to 1. Figure 2.24 shows a GT2FS taken from (Coupland & John 2008b), where  $\mu_{\tilde{A}}(x) \equiv U$  and  $\mu_{\tilde{A}}(x, u) \equiv \tilde{U}$  using the notations of this thesis. To perform operations using GT2FSs, methods from computational geometry are proposed for calculating the join and meet of T2FSs. These methods are shown to have reduced the computational complexity of computing the Centroid compared to using WSRT (John & Coupland 2007, Coupland & John 2007, Coupland & John 2008b). Mendel et al. (2009) argued that, “A limitation of the geometric and grid methods is that they do not obtain closed-form formulas for the join and meet operations”, these closed form formulas are desired in some situations. Another disadvantage that is particularly important for this thesis is the simplicity of extending operations. It is not easy to define an extension mechanism in order for functions and operations to be extended directly from known forms of classical fuzzy sets, interval valued fuzzy sets or crisp sets.

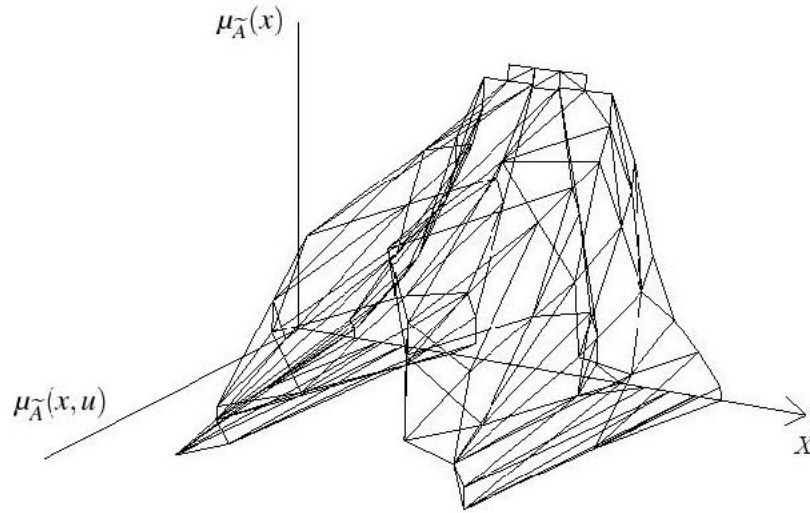


Fig. 2.24. Geometric T2FS taken from Coupland and John (2008).

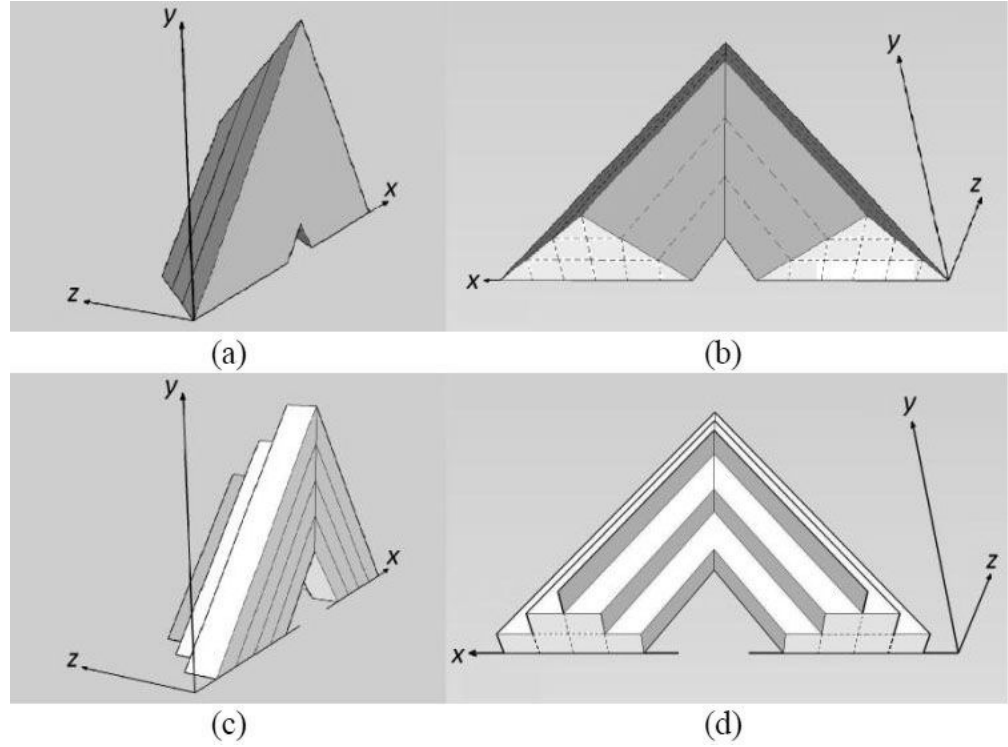


Fig. 2.25. zSlice based T2FS taken from Wagner and Hagrais (2008).

## 2.7 Zadeh's T2FSs: Revisited

This section revisits Zadeh's use of the EP for T2FS operations. The reason for this is to highlight the relationship between  $\alpha$ -cuts of FSs and T2FSs, which in-effect gives insight to the representations defined by Liu (2008) and Wagner & Hagrais (2008). It is summarised in two stages:

1. Extend the T1FS definition to fuzzy sets with interval-valued membership functions.
2. Generalise from intervals to fuzzy sets by the use of the  $\alpha$ -cut form of the EP ( $\alpha$ -EP).

Zadeh (1975a) demonstrated this procedure by defining the intersection of two T2FSs. First, let  $A$  and  $B$  be two FSs on  $X$ , and the intersection of these two sets defined as follows

$$A \cap B \Leftrightarrow (A \cap B)(x) = A(x) \wedge B(x); \forall x \in X. \quad (2.48)$$

Next, extend this definition as stated in stage (1) to interval-valued membership grades  $A(x) = [\underline{a}, \bar{a}]$  and  $B(x) = [\underline{b}, \bar{b}]$  then using interval operations

$$[\underline{a}, \bar{a}] \wedge [\underline{b}, \bar{b}] = [\underline{a} \wedge \underline{b}, \bar{a} \wedge \bar{b}] \quad (2.49)$$

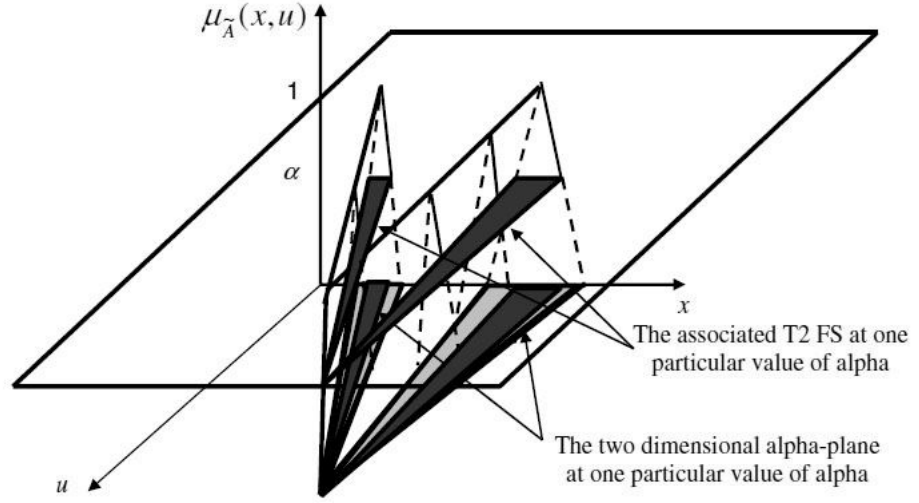


Fig. 2.26. Alpha-plane representation of T2FSs taken from Liu (2008).

Second, let  $\tilde{A}$  and  $\tilde{B}$  be two T2FSs on  $X$ , and their membership function (vertical slices) at each domain value are defined as follows:

$$\tilde{A}(x) \equiv \tilde{A}_x \Leftrightarrow \tilde{A}_x(a_x) \quad (2.50)$$

$$\tilde{B}(x) \equiv \tilde{B}_x \Leftrightarrow \tilde{B}_x(b_x) \quad (2.51)$$

then, take the  $\alpha$ -cuts of these vertical slices, in a continuous case this will appear as follows:

$$\tilde{A}_{x,\tilde{\alpha}} = \{a_x \mid \tilde{A}_x(a_x) \geq \tilde{\alpha}\} = [\underline{a}_{x,\tilde{\alpha}}, \bar{a}_{x,\tilde{\alpha}}] \quad (2.52)$$

$$\tilde{B}_{x,\tilde{\alpha}} = \{b_x \mid \tilde{B}_x(b_x) \geq \tilde{\alpha}\} = [\underline{b}_{x,\tilde{\alpha}}, \bar{b}_{x,\tilde{\alpha}}] \quad (2.53)$$

Then for each  $x$

$$(\tilde{A}_x \cap \tilde{B}_x)_{\tilde{\alpha}} = \tilde{A}_{x,\tilde{\alpha}} \wedge \tilde{B}_{x,\tilde{\alpha}}$$

and using the  $\alpha$ -EP produces

$$(\tilde{A} \cap \tilde{B})(x) = \bigcup_{\forall \tilde{\alpha}} \tilde{\alpha} (\tilde{A}_{x,\tilde{\alpha}} \wedge \tilde{B}_{x,\tilde{\alpha}}) \quad (2.54)$$

Up to this point Zadeh finalised his derivation. To generalise, it is clear that

$$\tilde{A} \cap \tilde{B} = \bigcup_{\forall x} (x, (\tilde{A} \cap \tilde{B})(x))$$

with the understanding that  $x$  is the domain value that corresponds to  $(\tilde{A} \cap \tilde{B})(x)$ . Then

$$\tilde{A} \cap \tilde{B} = \bigcup_{\forall x} \left( x, \bigcup_{\forall \tilde{\alpha}} \tilde{\alpha} (\tilde{A}_{x, \tilde{\alpha}} \wedge \tilde{B}_{x, \tilde{\alpha}}) \right)$$

## 2.8 Alpha-planes and zSlices

Recently, Chen & Kawase (2000), Tahayori et al. (2006), Liu (2008) and Mendel et al. (2009), and Wagner & Hagrass (2008) and Wagner & Hagrass (2010) focused their attention towards decomposing T2FSs into several IVFSs. In particular, Liu (2008) defined  $\alpha$ -planes and Wagner & Hagrass (2008) defined zSlices as part of their effort to calculate the Centroid of T2FSs.

**Definition 2.8.1 ( $\alpha$ -plane)** *The  $\alpha$ -plane,  $\tilde{A}_{\tilde{\alpha}}$ , of a T2FS,  $\tilde{A}$ , is defined by the union of elements and associated PGs of  $\tilde{A}$  whose SGs are greater than or equal to level  $\tilde{\alpha}$ , i.e.,*

$$\tilde{A}_{\tilde{\alpha}} = \left\{ (x, u_x) \mid \tilde{A}_x(u_x) \geq \tilde{\alpha}, \forall x, \forall u_x \in J_x \right\} \quad (2.55)$$

*Adapted from Liu (2008)*

The text of the definition provided by Liu (2008) and Mendel et al. (2009) is slightly different, but the equation is the same. In their definition they mention the union of PGs which is incorrect. Then Liu defined an indicator function,  $I_{\tilde{A}_{\tilde{\alpha}}}$ , acting on  $x \in X$  such that,

$$I_{\tilde{A}_{\tilde{\alpha}}}(x, u_x) = \begin{cases} 1, & (x, u_x) \in \tilde{A}_{\tilde{\alpha}} \\ 0, & (x, u_x) \notin \tilde{A}_{\tilde{\alpha}} \end{cases} \quad (2.56)$$

Then a T2FS associated with each  $\alpha$ -plane,  $\tilde{\alpha}\tilde{A}_{\tilde{\alpha}}$ , is defined.

**Definition 2.8.2** *A T2FS associated with each  $\alpha$ -plane,  $\tilde{\alpha}\tilde{A}_{\tilde{\alpha}}$ , of a T2FS  $\tilde{A}$  is defined as follows:*

$$\tilde{\alpha}\tilde{A}_{\tilde{\alpha}} = \left\{ \left( (x, u_x), \tilde{\alpha} \cdot I_{\tilde{A}_{\tilde{\alpha}}}(x, u_x) \right) \mid \forall x \in X \right\} \quad (2.57)$$

*Adapted from Liu (2008)*

Using this definition the T2FS,  $\tilde{A}$ , is represented by the union of all its associated T2FSs, i.e.,

$$\tilde{A} = \bigcup_{\forall \tilde{\alpha}} \tilde{\alpha}\tilde{A}_{\tilde{\alpha}} \quad (2.58)$$

Figure 2.26 shows a T2FS with an  $\alpha$ -plane and the associated T2FS. This figure is taken from Liu (2008) with  $x$  representing the domain of the T2FS,  $u$  representing the primary membership grades



domain denoted  $U$  in this thesis notation, and  $\mu_{\tilde{A}}(x, u)$  represents the secondary grades domain denoted  $\tilde{U}$  in this thesis. Earlier in the definition of  $\alpha$ -cuts for FSs the method that involves the

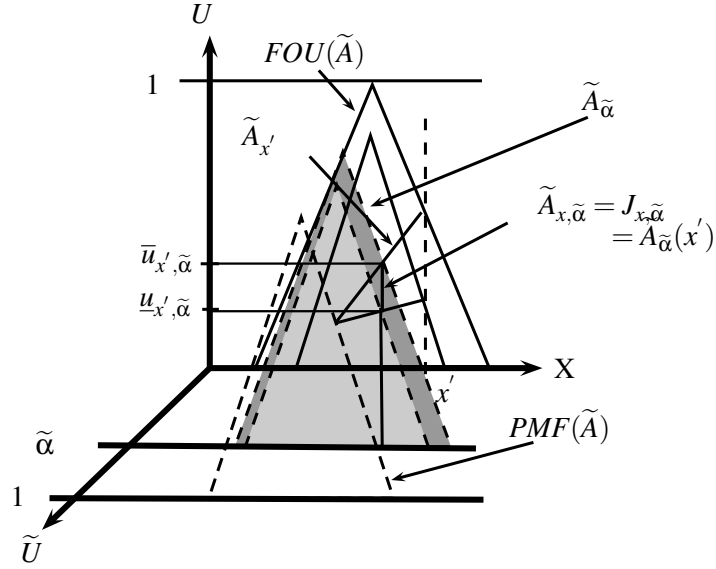


Fig. 2.27. The  $\alpha$ -plane  $\tilde{A}_{\tilde{\alpha}}$  of T2FS  $\tilde{A}$ .

indicator function is replaced with a rather simpler and straight forward notation, and likewise a similar trend is followed for the definition of  $\alpha$ -planes later in the thesis. On the other hand zSlices based T2FS representation is synonymous to the  $\alpha$ -plane representation. The concept is completely the same, and the zSlice based T2FS is defined with a slightly different notation of that used by the original version provided in Wagner & Hagrass (2008).

**Definition 2.8.3 (zSlice)** A zSlice is formed by slicing the general T2FS in the third dimension  $z$  (representing the secondary grades  $\tilde{A}_x(u_x)$ ) at level  $z_i$  (in other words  $\tilde{\alpha}$ ). The zSlice ( $\tilde{Z}_i$ ) is a T2FS with secondary membership grades fixed to  $z_i \in [0, 1]$ , and defined as follows:

$$\tilde{Z}_i = \bigcup_{\forall x} \bigcup_{u_{i_x} \in J_{i_x}} \{(x, u_{i_x}), z_i\}$$

where  $J_{i_x} = [u_{i_x}, \bar{u}_{i_x}]$  is the bounds of the interval formed by the slice  $z_i$  of the T2FS on the vertical slice at domain value  $x$ .

Figure 2.25 shows a T2FS with three zSlices. This figure is taken from Wagner & Hagrass (2008) with  $x$  representing the domain of the T2FS,  $y$  representing the primary membership grades domain denoted  $U$  in this thesis notation, and  $z$  represents the secondary grades domain denoted  $\tilde{U}$  in this thesis. This definition shares the same resemblance to the associated T2FS definition if the zSlice

level  $z_i$  is set to equate the  $\alpha$ -cut  $\tilde{\alpha}$ . The T2FS then is formed by the union of all the zSlices.

$$\tilde{A} = \bigcup_{\forall z_i \in (0,1]} \tilde{Z}_i \quad (2.59)$$

In this thesis only  $\alpha$ -planes are used and the same results are valid for zSlices. The reason for adapting the  $\alpha$ -plane formulation is the analogy it shares with the well known concept of  $\alpha$ -cuts in FSs. Moreover,  $\alpha$ -planes are developed using  $\alpha$ -cuts of VSs.

**Definition 2.8.4** *Let,  $\tilde{A}$  and  $\tilde{B}$ , be two T2FSs and,  $\tilde{A}_{\tilde{\alpha}}$  and  $\tilde{B}_{\tilde{\alpha}}$ , be their  $\alpha$ -planes at level  $\tilde{\alpha}$ . Then,*

1.  $(\tilde{A} \cup \tilde{B})_{\tilde{\alpha}} = \tilde{A}_{\tilde{\alpha}} \cup \tilde{B}_{\tilde{\alpha}}$ .
2.  $(\tilde{A} \cap \tilde{B})_{\tilde{\alpha}} = \tilde{A}_{\tilde{\alpha}} \cap \tilde{B}_{\tilde{\alpha}}$ .

*Adapted from Liu (2008)*

Mendel et al. (2010) provide a proof for the union of two T2FSs using  $\alpha$ -planes, and also show how to compute the intersection and Centroid. Wagner & Hagrass (2008) provide an alternative proof using VSs for the three operations. Some important facts about  $\alpha$ -planes are also observed in those papers:

- The FOU is equal to the  $\alpha$ -plane at level  $\tilde{0}$ , i.e.  $FOU(\tilde{A}) = \tilde{A}_{\tilde{0}}$ .
- Each  $\alpha$ -plane is characterised by an interval membership, in other words they are IVFSs. Wagner & Hagrass (2008) clearly state that, "a zSlice  $\tilde{Z}_i$  is equivalent to an interval type-2 fuzzy set with the exception that its membership grade  $\tilde{Z}_i(x, u_x)$  in the third dimension is not fixed to 1 but is equal to  $z_i$  where  $0 < z_i \leq 1$ ", the IT2FS in the statement are in fact the associated T2FSs (Mendel 2010). The connection between these concepts are clarified in the next subsection, it is used in Section 3.2 of Chapter 3 with the emphasis put on the development of the new  $\alpha$ -cut representation of T2FSs.
- Chen & Kawase (2000) stated that, "the conclusion of a fuzzy-valued fuzzy reasoning is obtained by an infinite number of interval-valued fuzzy reasoning". In this statement fuzzy-valued fuzzy sets are used to refer to T2FSs. Later in Section 3.2 a formula that represents this assertion in algebraic form is developed and proved. This formula will allow the extension of more generalised operations and functions not only the reasoning operations (i.e. the Join and Meet).
- Liu (2008) stated the following assertion as a property of  $\alpha$ -planes  $\tilde{\alpha}_i \geq \tilde{\alpha}_j \Rightarrow \tilde{A}_{\tilde{\alpha}_i} \subseteq \tilde{A}_{\tilde{\alpha}_j}$ . The nature of  $\tilde{A}_{\tilde{\alpha}_i} \subseteq \tilde{A}_{\tilde{\alpha}_j}$  has not been defined by Liu. Later in Section 3.4 of Chapter 3 some aspects of this property are investigated.

### 2.8.1 A discussion on $\alpha$ -planes

The origin of  $\alpha$ -planes is rooted in the definition of the intersection of two T2FSs introduced by Zadeh (1975a). It is customary to return to Zadeh's steps in order to decompose T2FSs to its elementary components, i.e. crisp sets. In general, since each VS is a FS, then it can be decomposed using the  $\alpha$ -cut decomposition theorem. Let  $\tilde{A} \in \tilde{F}(X)$  be a T2FS on  $X$ , where  $\tilde{A}_x$  is its VS at  $x$ . The  $\alpha$ -cuts of each VS are  $\tilde{A}_{x,\tilde{\alpha}} = \{u_x \mid \tilde{A}_x(u_x) \geq \tilde{\alpha}\}$ ,  $\forall u_x \in J_x$ . If the domain of the T2FS membership function is assumed to be continuous then  $\tilde{A}_{x,\tilde{\alpha}} = [\underline{u}_{x,\tilde{\alpha}}, \bar{u}_{x,\tilde{\alpha}}]$ . Since these VSs are FSs then they can be represented by the  $\alpha$ -cut decomposition theorem, i.e.,

$$\tilde{A}_x = \bigcup_{\forall \tilde{\alpha}} \tilde{\alpha} \tilde{A}_{x,\tilde{\alpha}} \quad (2.60)$$

where  $\tilde{\alpha} \tilde{A}_{x,\tilde{\alpha}}$  is the special FS ( $\alpha$ -FS) associated with each  $\alpha$ -cut. It is defined as  $\tilde{\alpha} \tilde{A}_{x,\tilde{\alpha}}(u_x) = \tilde{\alpha} \wedge \tilde{A}_{x,\tilde{\alpha}}(u_x)$  and  $\tilde{A}_{x,\tilde{\alpha}}(u_x) = 1$  if  $u_x \in \tilde{A}_{x,\tilde{\alpha}}$  and zero otherwise. Then, T2FS  $\tilde{A}$  is the union of all its VSs, therefore,

$$\tilde{A} = \bigcup_{\forall x} \left( x, \bigcup_{\forall \tilde{\alpha}} \tilde{\alpha} \tilde{A}_{x,\tilde{\alpha}} \right) \quad (2.61)$$

This is a very important result, indeed a T2FS is represented using a collection of crisp sets (or intervals) defined vertically. Now these crisp sets are manipulated in a meaningful way by taking the union of all the  $\alpha$ -cuts across all domain values for only one level, i.e.,  $\bigcup_{\forall x} (x, \tilde{A}_{x,\tilde{\alpha}})$ . It is the union of all the pairs  $(x, u_x)$  such that  $\tilde{A}_x(u_x) \geq \tilde{\alpha}$ . This is exactly the same as the  $\alpha$ -plane definition of Eq. 2.55.

$$\begin{aligned} \tilde{A}_{\tilde{\alpha}} &= \bigcup_{\forall x} (x, \tilde{A}_{x,\tilde{\alpha}}) \\ &= \left\{ (x, u_x) \mid \tilde{A}_x(u_x) \geq \tilde{\alpha}, \forall x, u_x \in J_x \right\} \end{aligned} \quad (2.62)$$

Here it is clear that

$$\tilde{A}_{\tilde{\alpha}}(x) = \tilde{A}_{x,\tilde{\alpha}} = J_{x,\tilde{\alpha}} \quad (2.63)$$

Then consider the  $\alpha$ -FSs of each VS by taking the union of all the  $\alpha$ -FSs across all domain values for only one level, i.e.,  $\bigcup_{\forall x} (x, \tilde{\alpha} \tilde{A}_{x,\tilde{\alpha}})$ . It is a T2FS with membership grades  $\tilde{\alpha} \tilde{A}_{x,\tilde{\alpha}}$ , which are FSs themselves, i.e.,  $\tilde{\alpha} \tilde{A}_{x,\tilde{\alpha}} = \bigcup_{\forall u_x} (u_x, \tilde{\alpha} \tilde{A}_{x,\tilde{\alpha}}(u_x))$ . This is exactly the same as the T2FS associated

with each  $\alpha$ -plane defined in Eq. 2.57.

$$\begin{aligned}
\tilde{\alpha}\tilde{A}_{\tilde{\alpha}} &= \bigcup_{\forall x} \left( x, \tilde{\alpha}\tilde{A}_{x,\tilde{\alpha}} \right) \\
&= \tilde{\alpha} \bigcup_{\forall x} \left( x, \tilde{A}_{x,\tilde{\alpha}} \right) \\
&= \left\{ \left( (x, u_x), \tilde{\alpha}\tilde{A}_{\tilde{\alpha}}(x, u_x) \right) \mid \forall x \in X \right\}
\end{aligned} \tag{2.64}$$

This is called a special T2FS associated with each  $\alpha$ -plane, ( $\alpha$ -**T2FS**), following the same convention used for FSs. It has to be mentioned that this same definition is called,  $\alpha$ -*FOU* in (Mendel et al. 2009), and *zSlice* in (Wagner & Hagrass 2008). Then it is clear to see that a T2FS is decomposed of these  $\alpha$ -T2FSs.

**Theorem 2.8.1 ( $\alpha$ -Plane RT)** *A type-2 fuzzy set,  $\tilde{A}$ , can be represented (decomposed) by the union of all its  $\alpha$ -T2FSs, i.e.,*

$$\tilde{A} = \bigcup_{\forall \tilde{\alpha}} \tilde{\alpha}\tilde{A}_{\tilde{\alpha}} \tag{2.65}$$

**Proof.** Straight forward from Eq. 2.61, Eq. 2.62 and Eq. 2.64. ■

In most cases  $\alpha$ -plane,  $\tilde{A}_{\tilde{\alpha}}$ , is considered to be an IVFS or an IT2FS (Liu 2008, Mendel et al. 2009, Wagner & Hagrass 2008, Wagner & Hagrass 2010). This fact is only true when the VSs are continuous functions and hence  $J_x \in \mathbf{I}(\mathbf{U})$ , i.e. it is an interval on  $[0, 1]$ . If the VSs are in discrete domains then as mentioned earlier, the PMs must be bounded through a bounding operation. An example of a bounding operation is taking the minimum and the maximum of the PGs of each PM. The following worked example demonstrates how to construct IVFS  $\alpha$ -planes for discrete T2FSs.

**Example 2.8.1** *Let  $X = \{x_i \mid i = 1, 2, \dots, 10\}$ , and very small(VS), small(S), medium(M), large(L), and very large(VL)  $\in \mathbf{F}(\mathbf{U})$  are the FSs that represent the vertical slices,  $\tilde{A}_x$ , defined in Table 2.2. Each vertical slice,  $\tilde{A}_{x_i}$ , consist of PGs,  $u_{x_i}$ , forming its domain and the SGs,  $\tilde{A}_{x_i}(u_{x_i})$ , forming its membership grade. Let also,  $\tilde{A} \in \tilde{\mathbf{F}}(\mathbf{X})$ , be defined as in Table 2.3, with domain values,  $x_i$ , corresponding to vertical slices from Table 2.2. Table 2.4 shows how to extract the  $\alpha$ -cuts ( $\tilde{\alpha}$ ) of the VS  $\tilde{A}_{x_i}$  of each domain value to form the crisp sets  $\tilde{A}_{x_i,\tilde{\alpha}}$ . Table 2.5 shows how to construct the interval membership grades of the  $\alpha$ -planes,  $\tilde{A}_{\tilde{\alpha}}(x_i) = \left[ \min \left( \tilde{A}_{x_i,\tilde{\alpha}} \right), \max \left( \tilde{A}_{x_i,\tilde{\alpha}} \right) \right]$  in order to formulate the IVFS  $\alpha$ -planes.*

Table 2.2. FSs that represent the vertical slices,  $\tilde{A}_x$ , in Example 2.8.1. The horizontal heading represents the SGs,  $\tilde{A}_x(u_x)$ , the vertical heading represents the VSs,  $\tilde{A}_x$ , and the numbers in between are the PGs,  $u_x$ .

$\tilde{A}_x$	0.0	0.5	1.0	0.5	0.0
VS	0.0	0.08	0.15	0.18	0.2
S	0.15	0.17	0.35	0.42	0.45
M	0.4	0.43	0.5	0.6	0.65
L	0.55	0.62	0.65	0.75	0.8
VL	0.7	0.78	0.85	0.9	1.0

Table 2.3. T2FS,  $\tilde{A}$ , in Example 2.8.1. Each domain value,  $x_i$ , along with its corresponding vertical slice from Table 2.2.

$x_i$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$	$x_9$	$x_{10}$
$\tilde{A}_{x_i}$	VS	VS	S	S	M	L	L	L	VL	VL

This example demonstrates the case when there are no gaps in the PM, i.e., all VSs are convex. If there is a contrary case, then these sets are approximated to an IVFS using a bounding operation such as taking the minimum and maximum (or infimum and supremum) of the PGs.

## 2.9 Discussion

The notations used in this thesis does not conform with the majority of publications on the topic of T2FSs. The fact that the notations adopted in this thesis are classic mathematical conventions support such a move. Lately, the issue of notation has been discussed by many researchers working in the type-2 fuzzy logic community (Coupland et al. 2010). In fact some researchers such as Aisbett et al. (2010) and Walker & Walker (2009) are advocating the use of classical mathematical standards rather than the type-2 fuzzy logic practices.

The variety of representations of the T2FSs is motivated by the quest for a practical method of performing operations in general and for calculating the Centroid of a T2FS in specific. To account for this assertion, Mendel & John (2002) mentioned that their WSRT focus on overcoming one particular difficulty of type-2 fuzzy logic:

“derivations of the formulas for the union, intersection, and complement of type-2 fuzzy sets all rely on using Zadehs Extension Principle, which in itself is a difficult concept (especially for newcomers to FL) and is somewhat ad hoc, so that deriving things using it may be considered problematic;...”

Mendel & John (2002) also stated another difficulty that they did not overcome using the WSRT:

Table 2.4. The crisp set  $\alpha$ -cuts,  $\tilde{A}_{x_i, \tilde{\alpha}}$ , of the vertical slices  $\tilde{A}_{x_i}$ , for each domain value,  $x_i$ , in Example 2.8.1.

$i$	$\tilde{\alpha} = 0.0$	$\tilde{\alpha} = 0.5$	$\tilde{\alpha} = 1.0$
1	0, 0.08, 0.15, 0.18, 0.2	0.08, 0.15, 0.18	0.15
2	0, 0.08, 0.15, 0.18, 0.2	0.08, 0.15, 0.18	0.15
3	0.15, 0.17, 0.35, 0.42, 0.45	0.17, 0.35, 0.42	0.35
4	0.15, 0.17, 0.35, 0.42, 0.45	0.17, 0.35, 0.42	0.35
5	0.4, 0.43, 0.5, 0.6, 0.65	0.43, 0.5, 0.6	0.5
6	0.55, 0.62, 0.65, 0.75, 0.8	0.62, 0.65, 0.75	0.65
7	0.55, 0.62, 0.65, 0.75, 0.8	0.62, 0.65, 0.75	0.65
8	0.55, 0.62, 0.65, 0.75, 0.8	0.62, 0.65, 0.75	0.65
9	0.7, 0.78, 0.85, 0.9, 1	0.78, 0.85, 0.9	0.85
10	0.7, 0.78, 0.85, 0.9, 1	0.78, 0.85, 0.9	0.85

Table 2.5. The interval membership grades of the  $\alpha$ -planes,  $\tilde{A}_{\tilde{\alpha}}(x_i)$  in Example 2.8.1.

$i$	$\tilde{\alpha} = 0.0$	$\tilde{\alpha} = 0.5$	$\tilde{\alpha} = 1.0$
1	[0, 0.2]	[0.08, 0.18]	[0.15, 0.15]
2	[0, 0.2]	[0.08, 0.18]	[0.15, 0.15]
3	[0.15, 0.45]	[0.17, 0.42]	[0.35, 0.35]
4	[0.15, 0.45]	[0.17, 0.42]	[0.35, 0.35]
5	[0.4, 0.65]	[0.43, 0.6]	[0.5, 0.5]
6	[0.55, 0.8]	[0.62, 0.75]	[0.65, 0.65]
7	[0.55, 0.8]	[0.62, 0.75]	[0.65, 0.65]
8	[0.55, 0.8]	[0.62, 0.75]	[0.65, 0.65]
9	[0.7, 1]	[0.78, 0.9]	[0.85, 0.85]
10	[0.7, 1]	[0.78, 0.9]	[0.85, 0.85]

“using type-2 fuzzy sets is computationally more complicated than using type-1 fuzzy sets.”

Coupland & John (2007) in the other hand articulated specifically what drives their work on geometric T2FSs:

“In the case of generalized type-2 membership functions where the secondary is a type-1 fuzzy number the computational complexity is very large. This has, in our view held back the exploitation of the real power of type-2 fuzzy systems”

Which in essence is trying to overcome that particular difficulty from Mendel & John (2002). Wagner & Hagrais (2008) stipulated their aim of defining zSlices in the abstract of their paper:

“The proposed approach will lead to a significant reduction in both the complexity and the computational requirements for general type-2 fuzzy logic systems. Hence, this will lead to facilitating the application of general type-2 fuzzy logic to many real world applications.”

Liu (2008) specifically mentioned the Centroid as the motivation for defining  $\alpha$ -planes:

“To date, the computation complexity of general type-2 fuzzy logic systems (T2 FLSs) is very high, which makes it very difficult to deploy them into practical applications”,..., “The objective of this paper is to develop an efficient strategy to implement centroid type-reduction for general T2 fuzzy logic systems.”

It is clear that both computational complexity and theoretical investigation formed much of the deriving force behind the recent developments on T2FSs. The newly developed  $\alpha$ -plane or zS-lice representations are yet to be used in applications, but they show a potential for reasonable performance.

## 2.10 Summary

This thesis is concerned with defining a novel representation of T2FSs that is useful both theoretically and practically. This chapter provides the foundation for the rest of the thesis to be built on. Specifically, this chapter discussed the terminology and notation used throughout the thesis for the main concepts of FSs, IVFSs, and T2FSs. Furthermore, some of the basic definitions and operations that will be utilised in the build-up of new concepts. In this chapter the different representations of T2FSs were reviewed, some of which are recently developed, e.g., the  $\alpha$ -plane and zSlices representations.

In the following chapter a theoretical investigation of  $\alpha$ -planes and IVFSs is conducted, and a new generalised extension principle is defined utilising the  $\alpha$ -plane RT. An alternative  $\alpha$ -cut representation of IVFSs is defined and some properties of these  $\alpha$ -planes are discussed in order to define the novel  $\alpha$ -cut representation theorem for T2FSs. In Chapter 4 and Chapter 5 this novel representation is shown to be useful for the theoretical investigation of measures of uncertainty and type-2 fuzzy numbers. Chapter 6 shows that this novel representation is suitable for practical implementations due its massively parallel nature.

## Chapter 3

# Alpha-cuts for Type-2 Fuzzy Sets

The  $\alpha$ -cut decomposition theorem is one of the fundamental concepts in the field of FSs. The power behind this theorem lies in the capability to decompose FSs into a collection of crisp sets. This decomposition along with the extension principle forms a methodology for extending mathematical concepts directly from crisp sets to FSs. In this chapter an new and novel mathematical formulations are presented, in which many theoretical investigations and practical applications are made possible. These novel mathematical formulations can be summarised in three main concepts:

1. The  $\alpha$ -plane extension principle. A novel generalisation is defined and a proof is provided, this generalisation allows functions and operations to be extended from IVFSs to T2FSs directly.
2. The  $\alpha$ -cut decomposition theorem for T2FSs. A novel representation theorem for T2FSs is presented, which decomposes T2FSs into a collection of crisp sets.
3. The  $\alpha$ -cut extension principle for T2FSs. Another novel generalisation is defined to allow functions and operations be extended from crisp sets to T2FSs in a straight forward mechanism.

Furthermore, other important definitions are presented in this chapter such as the core, support and containment of T2FSs. The  $\alpha$ -plane RT and the novel  $\alpha$ -cut RT provide alternative definitions for these particular concepts in the same manner that the  $\alpha$ -cut RT for FSs provides alternative definitions for the FS counterparts of these concepts. This chapter also introduces an alternative method of defining  $\alpha$ -cuts for IVFSs to those methods reviewed earlier in Section 2.9. Worked examples are used extensively to demonstrate the usefulness of the concepts defined in this chapter. These examples will include the use of the three main concepts listed above in defining the union and intersection of T2FSs. This chapter contains a set of interrelated novel definitions which together form one of the main contributions of this thesis, the derivation and proof of the  $\alpha$ -cut of T2FSs. Throughout this chapter all definitions and theorems are novel findings. On the other hand throughout the rest of the thesis, whenever there is a definition that is not the sole contribution of the thesis



it will be clearly shown in the title of the definition. Obviously if any definition or theorem is adapted it will be clearly mentioned. In short, all the material presented in this chapter is novel unless clearly stated otherwise.

### 3.1 Alpha-cuts of IVFSs: An Alternative Approach

In this section novel  $\alpha$ -cuts of IVFSs are investigated. The methods discussed are alternative approaches to the definitions provided in Section 2.9. The main difference is that in this chapter the  $\alpha$ -cuts are defined for the LMF and UMF independently. The reason behind this is, first, to allow the involvement of direct FS computation rather than the rearrangement introduced by Kaufmann & Gupta (1985) in Eq. 2.16. The rearrangement occurs after the  $\alpha$ -cut of the LMF and UMF are calculated, later in this section this difference is discussed. Second, the use of the LMF and UMF allow a direct connection with FS  $\alpha$ -cuts which will provide a platform for comparable theoretical studies unlike the methods by Zeng & Shi (2005) and Yager (2008a) shown in Section 2.9. Example 3.1.1 demonstrates the difference between both methods. In the following definition the  $\alpha$ -cuts of IVFSs are defined using the LMF and UMF:

**Definition 3.1.1 (IVFS  $\alpha$ -cuts)** *The  $\alpha$ -cut of an IVFS,  $\hat{A} \in X$ , is defined by taking the  $\alpha$ -cuts of its LMF and UMF at the same level  $\alpha$ , i.e.,*

$$\hat{A}_\alpha = (\underline{A}_\alpha, \bar{A}_\alpha)$$

where  $\hat{A}_\alpha(x) = [\underline{A}_\alpha(x), \bar{A}_\alpha(x)]$ .

In Definition 2.2.4 IVFSs are completely defined using two FSs, and in Definition 3.1.1 the  $\alpha$ -cuts of IVFSs are completely defined using the  $\alpha$ -cuts of these two FSs. In this case, the membership grade of each domain value,  $x$ , in the set,  $\hat{A}_\alpha$ , is an interval, i.e.,

$$\hat{A}_\alpha(x) = \begin{cases} [0, 0], & x \notin \underline{A}_\alpha \text{ and } x \notin \bar{A}_\alpha \\ [0, 1], & x \notin \underline{A}_\alpha \text{ and } x \in \bar{A}_\alpha \\ [1, 1], & x \in \underline{A}_\alpha \text{ and } x \in \bar{A}_\alpha \end{cases} \quad (3.1)$$

These situations are depicted in Figure 3.1. Notice that a particular impossible situation is not included, that of  $\hat{A}_\alpha(x) = [1, 0]$ . It means that there exist a domain value  $x'$  that belongs to the  $\alpha$ -cut of LMF and does not belong to the  $\alpha$ -cut of the UMF, which is not possible. By definition the LMF is always a subset of the UMF,  $\underline{A} \subseteq \bar{A}$ , i.e.,  $\underline{A}(x) \leq \bar{A}(x)$ ,  $\forall x$ . Based on Theorem 2.1.3 the  $\alpha$ -cut of the LMF is always a subset of the  $\alpha$ -cut of the UMF, i.e.,  $\underline{A}_\alpha \subseteq \bar{A}_\alpha$ ,  $\forall \alpha$ , which makes  $\hat{A}_\alpha(x) = [1, 0]$  impossible. Definition 3.1.1 shows that the  $\alpha$ -cuts of IVFSs are special IVFSs. These special IVFSs are bound by two distinct crisp sets, one that includes domain values that

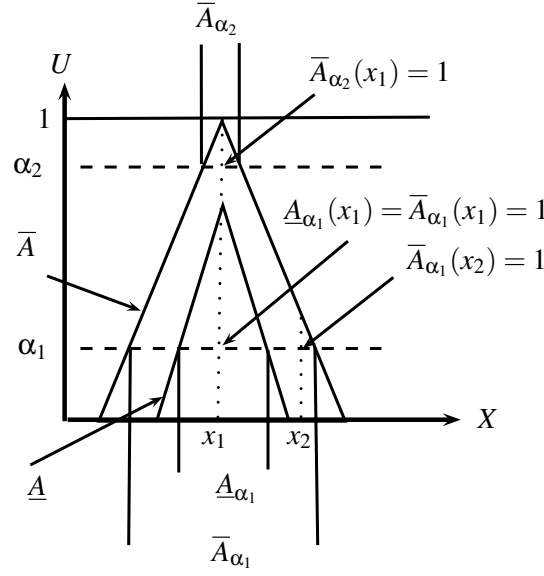


Fig. 3.1. IVFS  $\hat{A}$ , its LMF  $\underline{A}$ , its UMF  $\bar{A}$  and their  $\alpha$ -cuts.

belong to the  $\alpha$ -cut of the LMF and the other contains domain values that belong to the  $\alpha$ -cut of the UMF. The  $\alpha$ -cut of an IVFS is completely defined using these two crisp sets. This is actually a significant finding, because computation for the  $\alpha$ -cuts can be performed independently. This fact makes it very appealing and captures the semantics of the IVFS definition. The IVFS is actually a FS with uncertain (or unknown) membership, and this uncertainty is represented through an interval. The LMF and UMF represents this uncertainty with the interpretation that the FS is not exactly known, instead what is known are the FS bounds. Earlier in Definition 2.1.14, a special FS called the  $\alpha$ -FS is defined, which is associated with each  $\alpha$ -cut of a FS. Following the same convention a special IVFS is defined based on the  $\alpha$ -FS of the LMF and the UMF.

**Definition 3.1.2 ( $\alpha$ -IVFS)** A special IVFS,  $\alpha\hat{A}_\alpha \in \hat{F}(X)$ , called  $\alpha$ -IVFS is defined as follows:

$$\begin{aligned}\alpha\hat{A}_\alpha &= \alpha(\underline{A}_\alpha, \bar{A}_\alpha) \\ &= (\alpha\underline{A}_\alpha, \alpha\bar{A}_\alpha)\end{aligned}\tag{3.2}$$

where  $\alpha\hat{A}_\alpha(x) = \alpha \wedge [\underline{A}_\alpha(x), \bar{A}_\alpha(x)] = [\alpha \wedge \underline{A}_\alpha(x), \alpha \wedge \bar{A}_\alpha(x)]$ .

Here  $\alpha\hat{A}_\alpha$  is an IVFS, and each domain value,  $x$ , is associated with an interval membership grade,  $\alpha\hat{A}_\alpha(x) \in I(U)$ . Also  $\alpha\underline{A}_\alpha$  and  $\alpha\bar{A}_\alpha$  are both  $\alpha$ -FSs, which are respectively the LMF and UMF of  $\alpha$ -IVFS  $\alpha\hat{A}_\alpha$  as seen in Figure 3.3. Now, a decomposition theorem can be defined for IVFSs based on these  $\alpha$ -cuts.

**Theorem 3.1.1 (IVFS  $\alpha$ -cut RT)** An interval valued fuzzy set,  $\hat{A}$ , can be represented by the union

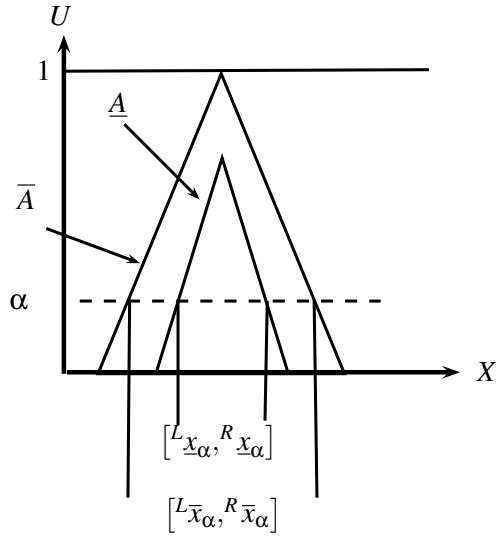


Fig. 3.2. Continuous IVFS  $\hat{A}$ , its LMF  $\underline{A}$ , its UMF  $\bar{A}$  and their  $\alpha$ -cuts.

of all its  $\alpha$ -IVFSs.

$$\hat{A} = \bigcup_{\forall \alpha} \alpha \hat{A}_\alpha \quad (3.3)$$

**Proof.** By definition any IVFS is represented using the LMF and UMF, i.e.,  $\hat{A} = (\underline{A}, \bar{A})$ . Since  $\underline{A} = \bigcup_{\forall \alpha} \alpha \underline{A}_\alpha$  and  $\bar{A} = \bigcup_{\forall \alpha} \alpha \bar{A}_\alpha$  by the decomposition theorem of FSs, then,

$$\begin{aligned} \hat{A} &= \left( \bigcup_{\forall \alpha} \alpha \underline{A}_\alpha, \bigcup_{\forall \alpha} \alpha \bar{A}_\alpha \right) \\ &= \bigcup_{\forall \alpha} (\alpha \underline{A}_\alpha, \alpha \bar{A}_\alpha) \end{aligned} \quad (3.4)$$

Straight forward from Definition 3.1.2  $\alpha \hat{A}_\alpha = (\alpha \underline{A}_\alpha, \alpha \bar{A}_\alpha)$ , and that completes the proof. ■

The following worked example demonstrates how to calculate the  $\alpha$ -cuts of discrete IVFSs.

**Example 3.1.1** Let  $X = \{x_i \mid i = 1, 2, \dots, 10\}$ , and  $\hat{A} \in \hat{F}(X)$  is an IVFS defined in Table 3.1. Table 3.2 shows the  $\alpha$ -cuts of IVFS  $\hat{A}$  calculated from its LMF and UMF. Table 3.3 shows how to reconstruct IVFS  $\hat{A}$  knowing its  $\alpha$ -cuts.

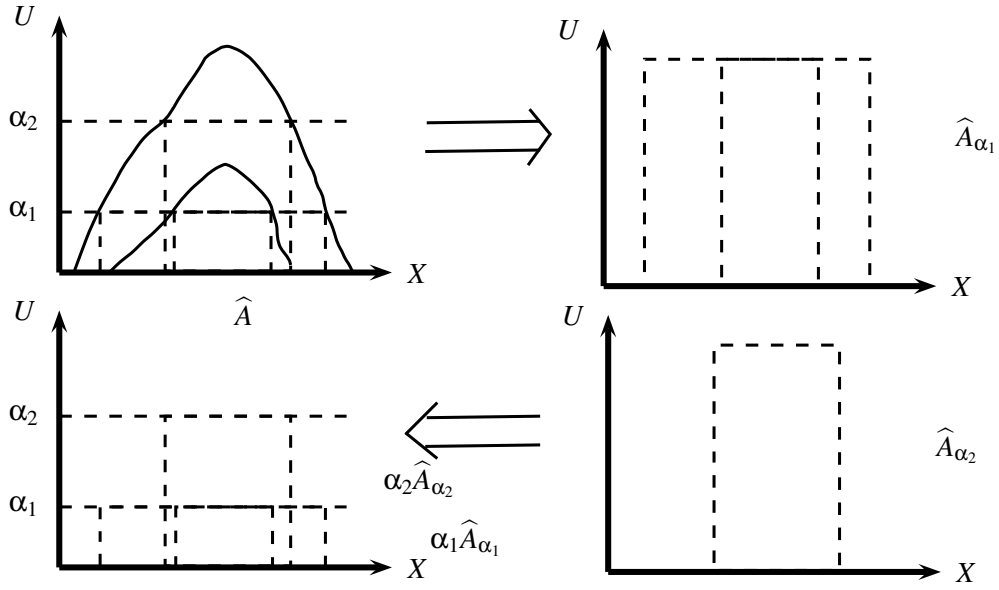


Fig. 3.3. The  $\alpha$ -IVFSs  $\alpha\hat{A}_\alpha$  of IVFS  $\hat{A}$ .

Using Eq. 3.3, if  $\hat{A}$  is a continuous and convex IVFS i.e.  $\underline{A}$  and  $\bar{A}$  are continuous and convex as seen in Figure 3.2, then an alternative approach to that of Eq. 2.16 can be defined.

**Definition 3.1.3** Let,  $\hat{A} \in \hat{F}(X)$  be a continuous and convex IVFS with LMF  $\underline{A}$  and UMF  $\bar{A}$ . Let also its  $\alpha$ -cut at level  $\alpha$  be  $\hat{A}_\alpha = (\underline{A}_\alpha, \bar{A}_\alpha)$  where  $\underline{A}_\alpha = [{}^L\underline{x}_\alpha, {}^R\underline{x}_\alpha]$  and  $\bar{A}_\alpha = [{}^L\bar{x}_\alpha, {}^R\bar{x}_\alpha]$ . Then,  $\hat{A}_\alpha$  can be calculated using the following formula:

$$\hat{A}_\alpha = \begin{cases} ([{}^L\underline{x}_\alpha, {}^R\underline{x}_\alpha], [{}^L\bar{x}_\alpha, {}^R\bar{x}_\alpha]) & , \alpha \leq h(\underline{A}) \text{ and } \alpha \leq h(\bar{A}) \\ (\emptyset, [{}^L\bar{x}_\alpha, {}^R\bar{x}_\alpha]) & , \alpha > h(\underline{A}) \text{ and } \alpha \leq h(\bar{A}) \\ (\emptyset, \emptyset) & , \alpha > h(\bar{A}) \end{cases} \quad (3.5)$$

where  $\forall \alpha : {}^L\bar{x}_\alpha \leq {}^L\underline{x}_\alpha \leq {}^R\underline{x}_\alpha \leq {}^R\bar{x}_\alpha$ ,  $h(\underline{A})$  is the height of LMF,  $h(\bar{A})$  is the height of UMF, and  $\emptyset$  is an empty set.

Earlier in Eq. 2.16 another method of defining  $\alpha$ -cuts for continuous and convex IVFSs have been defined. There are two drawbacks to this method, first, it does not reduce to the  $\alpha$ -cut of FSs directly, instead some manipulation and rearrangement must be done. To explain this case assume that  $\hat{A} \in \mathbf{F}(X)$  which means that both the LMF and UMF are equal i.e.  $\underline{A} = \bar{A}$ . Then,  ${}^L\bar{x}_\alpha = {}^L\underline{x}_\alpha = {}^Lx_\alpha$  and  ${}^R\underline{x}_\alpha = {}^R\bar{x}_\alpha = {}^Rx_\alpha$ . Then according to Eq. 2.16 the  $\alpha$ -cut  $\hat{A}_\alpha$  of  $\hat{A}$  is  $\{[{}^Lx_\alpha, {}^Lx_\alpha], [{}^Rx_\alpha, {}^Rx_\alpha]\}$  which can be simplified to  $\{{}^Lx_\alpha, {}^Rx_\alpha\}$ . This form does not reduce

Table 3.1. IVFS,  $\hat{A}$ , in Example 3.1.1. Each domain value,  $x_i$ , along with its corresponding interval membership grade, LMF membership grade and UMF membership grade.

$x_i$	$\hat{A}(x_i)$	$\underline{A}(x_i)$	$\bar{A}(x_i)$
$x_1$	[0, 0.6]	0	0.6
$x_2$	[0, 0.8]	0	0.8
$x_3$	[0, 0.9]	0	0.9
$x_4$	[0.5, 1]	0.5	1
$x_5$	[0.7, 1]	0.7	1
$x_6$	[0.6, 1]	0.6	1
$x_7$	[0.3, 0.8]	0.3	0.8
$x_8$	[0, 0.6]	0	0.6
$x_9$	[0, 0.3]	0	0.3
$x_{10}$	[0, 0.1]	0	0.1

Table 3.2. The  $\alpha$ -cuts of IVFS,  $\hat{A}$ , of Table 3.1 in Example 3.1.1.

$\alpha$	$\underline{A}_\alpha$	$\bar{A}_\alpha$
0.0	$\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$	$\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$
0.1	$\{x_4, x_5, x_6, x_7\}$	$\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$
0.2	$\{x_4, x_5, x_6, x_7\}$	$\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$
0.3	$\{x_4, x_5, x_6, x_7\}$	$\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$
0.4	$\{x_4, x_5, x_6\}$	$\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$
0.5	$\{x_4, x_5, x_6\}$	$\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$
0.6	$\{x_5, x_6\}$	$\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$
0.7	$\{x_5\}$	$\{x_2, x_3, x_4, x_5, x_6, x_7\}$
0.8	$\emptyset$	$\{x_2, x_3, x_4, x_5, x_6, x_7\}$
0.9	$\emptyset$	$\{x_3, x_4, x_5, x_6\}$
1.0	$\emptyset$	$\{x_4, x_5, x_6\}$

in a natural manner to the  $\alpha$ -cut of a FS represented by  $[^Lx_\alpha, ^Rx_\alpha]$  although the end points are preserved, but the meaning is not the same in the sense that  $\{\cdot, \cdot\} \neq [\cdot, \cdot]$ . On the other hand, according to Eq. 3.5 the  $\alpha$ -cut  $\hat{A}_\alpha$  of  $\hat{A}$  is  $([^Lx_\alpha, ^Rx_\alpha], [^Lx_\alpha, ^Rx_\alpha])$  which is compatible with the definition of IVFS being represented by two FSs. When these two fuzzy sets are equal the  $\alpha$ -cut of the resulting FS is equal to either  $\alpha$ -cut of the LMF or the UMF. The second drawback is it does not hold the semantics of  $\alpha$ -cuts through out the representation. To explain this, in Eq. 2.16, what does  $x \in [^L\bar{x}_\alpha, ^Lx_\alpha]$  represent? It has a rather complicated relationship to LMF and UMF. It is the values  $x$  of the domain that belongs to  $\bar{A}_\alpha$  and does not belong to  $\underline{A}_\alpha$  except its bounds! this can mathematically be described as follows:

$$\begin{aligned}\hat{A}_\alpha &= \{x \mid x \in [^L\bar{x}_\alpha, ^Rx_\alpha] \text{ and } x \notin (^Lx_\alpha, ^Rx_\alpha)\} \\ &= \bar{A}_\alpha \cap \underline{A}'_{\alpha^+}\end{aligned}\tag{3.6}$$

where  $(^Lx_\alpha, ^Rx_\alpha)$  is an interval with excluded endpoints, and  $\underline{A}'_{\alpha^+}$  is the complement of the strong  $\alpha$ -cut ( $\alpha^+$ ) of the LMF  $\underline{A}$ . On the other hand, using Eq. 3.5 any domain value  $x$  either belongs to

Table 3.3. Regenerating IVFS,  $\widehat{A}$ , in Example 3.1.1 from its  $\alpha$ -cuts in Table 3.2.

$i$	$\alpha A_\alpha(x_i)$	$\alpha \bar{A}_\alpha(x_i)$	$\widehat{A}(x_i)$
1	0	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6	[0, 0.6]
2	0	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8	[0, 0.8]
3	0	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9	[0, 0.9]
4	0, 0.1, 0.2, 0.3, 0.4, 0.5	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1	[0.5, 1]
5	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1	[0.7, 1]
6	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1	[0.6, 1]
7	0, 0.1, 0.2, 0.3	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8	[0.3, 0.8]
8	0	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6	[0, 0.6]
9	0	0, 0.1, 0.2, 0.3	[0, 0.3]
10	0	0, 0.1	[0, 0.1]

both the  $\alpha$ -cuts of the LMF and the UMF, belongs to the  $\alpha$ -cut of the UMF and does not belong to the  $\alpha$ -cut of the LMF, or does not belong to both  $\alpha$ -cuts. Clearly, preserving the LMF and UMF independance simplifies calculations and preserves the meaning of the operations conducted on IVFSs. Although there are different ways to define  $\alpha$ -cuts for IVFSs, the representation theorem is the same. The ability to extend operations using the  $\alpha$ -cut RT is what makes it useful, and the following theorem defines an  $\alpha$ -EP comparable to that of FSs keeping in mind that the LMF and UMF are also preserved distinct from each other.

**Theorem 3.1.2 (IVFS  $\alpha$ -EP)** *Let,  $X = X_1 \times \dots \times X_n$ , be the Cartesian product of universes, and  $\widehat{A}_1, \dots, \widehat{A}_n$  be IVFSs in each universe respectively. Also let  $Y$  be another universe and  $\widehat{B} \in Y$  be an IVFS such that  $\widehat{B} = f(\widehat{A}_1, \dots, \widehat{A}_n)$ , where  $f : X \rightarrow Y$  is a monotonic mapping. Then,  $\widehat{B}$ , is equal to the union of applying the same function to all the decomposed  $\alpha$ -cuts of the IVFSs (Hamrawi et al. 2010), i.e.,*

$$\begin{aligned}
 \widehat{B} &= f(\widehat{A}_1, \dots, \widehat{A}_n) \\
 &= \bigcup_{\forall \alpha} \alpha \left( f(\widehat{A}_{1\alpha}, \dots, \widehat{A}_{n\alpha}) \right) \\
 &= \bigcup_{\forall \alpha} \alpha \left( f(\underline{A}_{1\alpha}, \dots, \underline{A}_{n\alpha}), f(\bar{A}_{1\alpha}, \dots, \bar{A}_{n\alpha}) \right)
 \end{aligned} \tag{3.7}$$

**Proof.** Since  $\underline{A}_1, \dots, \underline{A}_n, \bar{A}_1, \dots, \bar{A}_n \in F(X)$ , then from Eq. 2.6

$$f(\underline{A}_1, \dots, \underline{A}_n) = \bigcup_{\forall \alpha} \alpha f(\underline{A}_{1\alpha}, \dots, \underline{A}_{n\alpha}) \tag{3.8}$$

$$f(\bar{A}_1, \dots, \bar{A}_n) = \bigcup_{\forall \alpha} \alpha f(\bar{A}_{1\alpha}, \dots, \bar{A}_{n\alpha}) \tag{3.9}$$

From Definition 2.2.6, the EP of IVFSs clearly states that

$$f(\hat{A}_1, \dots, \hat{A}_n) = (f(\underline{A}_1, \dots, \underline{A}_n), f(\bar{A}_1, \dots, \bar{A}_n)) \quad (3.10)$$

Then substitute Eq. 3.8 and Eq. 3.9 in Eq. 3.10 to arrive at

$$\begin{aligned} f(\hat{A}_1, \dots, \hat{A}_n) &= \left( \bigcup_{\forall \alpha} \alpha f(\underline{A}_{1\alpha}, \dots, \underline{A}_{n\alpha}), \bigcup_{\forall \alpha} \alpha f(\bar{A}_{1\alpha}, \dots, \bar{A}_{n\alpha}) \right) \\ &= \bigcup_{\forall \alpha} \alpha (f(\underline{A}_{1\alpha}, \dots, \underline{A}_{n\alpha}), f(\bar{A}_{1\alpha}, \dots, \bar{A}_{n\alpha})) \end{aligned}$$

which completes the proof. ■

The following example shows how to perform the union and intersection of IVFSs using  $\alpha$ -cuts.

**Example 3.1.2** Let  $\hat{4}$  and  $\hat{8}$  be two IVFS defined in Table 3.4 and Table 3.5, respectively. The  $\alpha$ -cuts of both their LMF and UMF is shown in Table 3.6. The union of the  $\alpha$ -cuts are shown in Table 3.7. This will eventually lead to an IVFS  $\hat{4} \cup \hat{8}$ . The method used to generate the membership grades of  $\hat{4} \cup \hat{8}$  from its  $\alpha$ -cuts is shown in Table 3.8. The intersection of the  $\alpha$ -cuts are shown in Table 3.9. This will eventually lead to an IVFS  $\hat{4} \cap \hat{8}$ . The method used to generate the membership grades of  $\hat{4} \cap \hat{8}$  from its  $\alpha$ -cuts is shown in Table 3.10.

Table 3.4. IVFS,  $\hat{4}$ , in Example 3.1.2.

$x$	2	3	4	5	6
$\hat{4}(x)$	[0, 0.2]	[0.4, 0.6]	[0.8, 1]	[0.5, 0.6]	[0, 0.4]

Table 3.5. IVFS,  $\hat{8}$ , in Example 3.1.2.

$x$	5	6	7	8	9	10	11
$\hat{8}(x)$	[0, 0.1]	[0.2, 0.5]	[0.6, 0.8]	[1, 1]	[0.5, 0.8]	[0.2, 0.4]	[0, 0.1]

Table 3.6. The  $\alpha$ -cuts of IVFS,  $\hat{4}$  and  $\hat{8}$ , in Example 3.1.2.

$\alpha$	$\underline{4}_\alpha$	$\underline{8}_\alpha$	$\overline{4}_\alpha$	$\overline{8}_\alpha$
0.0	$\{2, 3, 4, 5, 6\}$	$\{5, 6, 7, 8, 9, 10, 11\}$	$\{2, 3, 4, 5, 6\}$	$\{5, 6, 7, 8, 9, 10, 11\}$
0.1	$\{3, 4, 5\}$	$\{6, 7, 8, 9, 10\}$	$\{2, 3, 4, 5, 6\}$	$\{5, 6, 7, 8, 9, 10, 11\}$
0.2	$\{3, 4, 5\}$	$\{6, 7, 8, 9, 10\}$	$\{2, 3, 4, 5, 6\}$	$\{6, 7, 8, 9, 10\}$
0.3	$\{3, 4, 5\}$	$\{7, 8, 9\}$	$\{3, 4, 5, 6\}$	$\{6, 7, 8, 9, 10\}$
0.4	$\{3, 4, 5\}$	$\{7, 8, 9\}$	$\{3, 4, 5, 6\}$	$\{6, 7, 8, 9, 10\}$
0.5	$\{4, 5\}$	$\{7, 8, 9\}$	$\{3, 4, 5\}$	$\{6, 7, 8, 9\}$
0.6	$\{4\}$	$\{7, 8\}$	$\{3, 4, 5\}$	$\{7, 8, 9\}$
0.7	$\{4\}$	$\{8\}$	$\{4\}$	$\{7, 8, 9\}$
0.8	$\{4\}$	$\{8\}$	$\{4\}$	$\{7, 8, 9\}$
0.9	$\emptyset$	$\{8\}$	$\{4\}$	$\{8\}$
1.0	$\emptyset$	$\{8\}$	$\{4\}$	$\{8\}$

Table 3.7. The  $\alpha$ -cuts of IVFS,  $\hat{4} \cup \hat{8}$ , in Example 3.1.2.

$\alpha$	$\underline{4}_\alpha \cup \underline{8}_\alpha$	$\overline{4}_\alpha \cup \overline{8}_\alpha$
0.0	$\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$	$\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$
0.1	$\{3, 4, 5, 6, 7, 8, 9, 10\}$	$\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$
0.2	$\{3, 4, 5, 6, 7, 8, 9, 10\}$	$\{2, 3, 4, 5, 6, 7, 8, 9, 10\}$
0.3	$\{3, 4, 5, 7, 8, 9\}$	$\{3, 4, 5, 6, 7, 8, 9, 10\}$
0.4	$\{3, 4, 5, 7, 8, 9\}$	$\{3, 4, 5, 6, 7, 8, 9, 10\}$
0.5	$\{4, 5, 7, 8, 9\}$	$\{3, 4, 5, 6, 7, 8, 9\}$
0.6	$\{4, 7, 8\}$	$\{3, 4, 5, 7, 8, 9\}$
0.7	$\{4, 8\}$	$\{4, 7, 8, 9\}$
0.8	$\{4, 8\}$	$\{4, 7, 8, 9\}$
0.9	$\{8\}$	$\{4, 8\}$
1.0	$\{8\}$	$\{4, 8\}$

To summarise the overall picture, the process of deriving operations for IVFSs can be viewed as defining these operations for two distinct FSs, i.e., the UMF and LMF. The same operations can be defined for crisp sets (or intervals) and then extend them to FSs using the  $\alpha$ -EP. The novel net step presented in this theorem is to define these operations for IVFSs by taking both FSs and using the  $\alpha$ -EP. To derive operations for IVFSs in such a simple and elegant process is in itself a significant result. The main motivation for defining the  $\alpha$ -cut of IVFSs is the significance of this theorem to the definition of the  $\alpha$ -cut RT for T2FSs.

### 3.2 T2FS Alpha-plane Extension Principle

This section introduces a novel generalisation that extends operations from IVFSs to T2FSs directly using  $\alpha$ -planes. This generalisation also plays the role of the foundation for the  $\alpha$ -cut RT for T2FSs, discussed later in Section 3.3. Early versions of this method has been stated without



Table 3.8. IVFS,  $\widehat{4} \cup \widehat{8}$ , in Example 3.1.2 from its  $\alpha$ -cuts in Table 3.7.

$x$	$\alpha(\underline{4} \cup \underline{8})_\alpha(x)$	$\alpha(\overline{4} \cup \overline{8})_\alpha(x)$	$(\widehat{4} \cup \widehat{8})(x)$
2	0	0, 0.1, 0.2	[0, 0.2]
3	0, 0.1, 0.2, 0.3, 0.4	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6	[0.4, 0.6]
4	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1	[0.8, 1]
5	0, 0.1, 0.2, 0.3, 0.4, 0.5	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6	[0.5, 0.6]
6	0, 0.1, 0.2	0, 0.1, 0.2, 0.3, 0.4, 0.5	[0.2, 0.5]
7	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8	[0.6, 0.8]
8	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1	[1, 1]
9	0, 0.1, 0.2, 0.3, 0.4, 0.5	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8	[0.5, 0.8]
10	0, 0.1, 0.2	0, 0.1, 0.2, 0.3, 0.4	[0.2, 0.4]
11	0	0, 0.1	[0, 0.1]

Table 3.9. The  $\alpha$ -cuts of IVFS,  $\widehat{4} \cap \widehat{8}$ , in Example 3.1.2.

$\alpha$	$\underline{4}_\alpha \cap \underline{8}_\alpha$	$\overline{4}_\alpha \cap \overline{8}_\alpha$
0.0	{5, 6}	{5, 6}
0.1	$\emptyset$	{5, 6}
0.2	$\emptyset$	{6}
0.3	$\emptyset$	{6}
0.4	$\emptyset$	{6}
0.5	$\emptyset$	$\emptyset$
0.6	$\emptyset$	$\emptyset$
0.7	$\emptyset$	$\emptyset$
0.8	$\emptyset$	$\emptyset$
0.9	$\emptyset$	$\emptyset$
1.0	$\emptyset$	$\emptyset$

a proof by Hamrawi & Coupland (2009b), and with a proof by Hamrawi et al. (2009). Recently Hamrawi et al. (2010) introduced some refinements that are included within the discussion below. The new generalised  $\alpha$ -plane EP for T2FSs is based on the  $\alpha$ -plane RT discussed earlier in Chapter 2. These  $\alpha$ -planes are or required to be IVFSs for practical reasons.

**Theorem 3.2.1 ( $\alpha$ -plane EP)** *Let,  $X = X_1 \times \dots \times X_n$ , be the Cartesian product of universes, and  $\widetilde{A}_1, \dots, \widetilde{A}_n$  be T2FSs in each universe respectively. Also let  $Y$  be another universe and  $\widetilde{B} \in Y$  be a T2FS such that  $\widetilde{B} = f(\widetilde{A}_1, \dots, \widetilde{A}_n)$ , where  $f : X \rightarrow Y$  is a monotonic mapping. Assume that all the decomposed  $\alpha$ -planes of all the T2FSs (i.e.  $\widetilde{A}_1, \dots, \widetilde{A}_n$ ) are or required to be IVFSs. Then  $\widetilde{B}$  is equal to the union of applying the same function to all the decomposed  $\alpha$ -planes of  $\widetilde{A}_1, \dots, \widetilde{A}_n$ , i.e.,*

$$\begin{aligned}
 \widetilde{B} &= f(\widetilde{A}_1, \dots, \widetilde{A}_n) \\
 &= \bigcup_{\forall \widetilde{\alpha}} \widetilde{\alpha} f(\widetilde{A}_{1, \widetilde{\alpha}}, \dots, \widetilde{A}_{n, \widetilde{\alpha}})
 \end{aligned} \tag{3.11}$$

Table 3.10. IVFS,  $\widehat{4} \cap \widehat{8}$ , in Example 3.1.2 from its  $\alpha$ -cuts in Table 3.9

$x$	$\alpha(\underline{4} \cap \underline{8})_\alpha(x)$	$\alpha(\overline{4} \cap \overline{8})_\alpha(x)$	$(\widehat{4} \cap \widehat{8})(x)$
5	0	0, 0.1	$[0, 0.1]$
6	0	0, 0.1, 0.2, 0.3, 0.4	$[0, 0.4]$

**Proof.** Starting from the EP for T2FSs,

$$\begin{aligned}
 \widetilde{B}(y) &= f(\widetilde{A}_1, \dots, \widetilde{A}_n)(y) \\
 &= \sup_{(x_1, \dots, x_n) = f^{-1}(y)} \min(\widetilde{A}_1(x_1), \dots, \widetilde{A}_n(x_n)) \\
 &= \sup_{(x_1, \dots, x_n) = f^{-1}(y)} \min(\widetilde{A}_{1,x_1}, \dots, \widetilde{A}_{n,x_n})
 \end{aligned} \tag{3.12}$$

since  $\widetilde{A}_{1,x_1}, \dots, \widetilde{A}_{n,x_n} \in F(X)$  then from Eq. 2.61,

$$\widetilde{A}_{i,x_i} = \bigcup_{\forall \widetilde{\alpha}} \widetilde{\alpha} \widetilde{A}_{i,x_i,\widetilde{\alpha}} \tag{3.13}$$

where  $i = 1, \dots, n$ . Then substituting this equation in Eq. 3.12 it yields to

$$\begin{aligned}
 \widetilde{B}(y) &= \sup_{(x_1, \dots, x_n) = f^{-1}(y)} \min \left( \bigcup_{\forall \widetilde{\alpha}} \widetilde{\alpha} \widetilde{A}_{1,x_1,\widetilde{\alpha}}, \dots, \bigcup_{\forall \widetilde{\alpha}} \widetilde{\alpha} \widetilde{A}_{n,x_n,\widetilde{\alpha}} \right) \\
 &= \sup_{(x_1, \dots, x_n) = f^{-1}(y)} \bigcup_{\forall \widetilde{\alpha}} \widetilde{\alpha} \min(\widetilde{A}_{1,x_1,\widetilde{\alpha}}, \dots, \widetilde{A}_{n,x_n,\widetilde{\alpha}}) \\
 &= \bigcup_{\forall \widetilde{\alpha}} \widetilde{\alpha} \sup_{(x_1, \dots, x_n) = f^{-1}(y)} \min(\widetilde{A}_{1,x_1,\widetilde{\alpha}}, \dots, \widetilde{A}_{n,x_n,\widetilde{\alpha}})
 \end{aligned} \tag{3.14}$$

now, since  $\widetilde{A}_{1,\widetilde{\alpha}}, \dots, \widetilde{A}_{n,\widetilde{\alpha}} \in \widehat{F}(X)$ , then

$$f(\widetilde{A}_{1,\widetilde{\alpha}}, \dots, \widetilde{A}_{n,\widetilde{\alpha}}) = \sup_{(x_1, \dots, x_n) = f^{-1}(y)} \min(\widetilde{A}_{1,\widetilde{\alpha}}(x_1), \dots, \widetilde{A}_{n,\widetilde{\alpha}}(x_n))$$

then, take the union of all  $\widetilde{\alpha}$ , i.e.,

$$\bigcup_{\forall \widetilde{\alpha}} \widetilde{\alpha} f(\widetilde{A}_{1,\widetilde{\alpha}}, \dots, \widetilde{A}_{n,\widetilde{\alpha}}) = \bigcup_{\forall \widetilde{\alpha}} \widetilde{\alpha} \sup_{(x_1, \dots, x_n) = f^{-1}(y)} \min(\widetilde{A}_{1,\widetilde{\alpha}}(x_1), \dots, \widetilde{A}_{n,\widetilde{\alpha}}(x_n)) \tag{3.15}$$

observe that  $\widetilde{A}_{i,\widetilde{\alpha}}(x_i) = \widetilde{A}_{i,x_i,\widetilde{\alpha}}, \forall i$ , it follows that Eq. 3.14 and Eq. 3.15 are equal, and that completes the proof. ■

The union and the intersection of T2FSs in Definition 2.8.4 by Liu (2008) and Wagner & Hagrass

(2008) can be derived using this theorem. Take for example the binary version of Eq. 3.11, i.e.,

$$f(\tilde{A}, \tilde{B}) = \bigcup_{\forall \tilde{\alpha}} \tilde{\alpha} f(\tilde{A}_{\tilde{\alpha}}, \tilde{B}_{\tilde{\alpha}}) \quad (3.16)$$

and let  $f$  be the union or intersection, i.e.,

$$\tilde{A} \cup \tilde{B} = \bigcup_{\forall \tilde{\alpha}} \tilde{\alpha} (\tilde{A}_{\tilde{\alpha}} \cup \tilde{B}_{\tilde{\alpha}}) \quad (3.17)$$

$$\tilde{A} \cap \tilde{B} = \bigcup_{\forall \tilde{\alpha}} \tilde{\alpha} (\tilde{A}_{\tilde{\alpha}} \cap \tilde{B}_{\tilde{\alpha}}) \quad (3.18)$$

It is clear that the standard union and intersection operations between IVFSs can be used to calculate the union and intersection between T2FSs. Example 3.2.1 shows the union of two T2FSs using the well known union calculation between IVFSs.

**Example 3.2.1** Consider the T2FSs,  $\tilde{3}$ , in Table 3.11 \* and,  $\tilde{6}$ , in Table 3.12. To perform the join, a decomposition of each T2FS into its  $\alpha$ -planes must be performed. The interval membership grades of each  $\alpha$ -plane are constructed using the bounds of the PMs  $J_{x, \tilde{\alpha}}$ . Table 3.13 shows the  $\alpha$ -planes of T2FS  $\tilde{3}$  and Table 3.14 shows the  $\alpha$ -planes of T2FS  $\tilde{6}$ . The union and intersection between the  $\alpha$ -planes of each level  $\tilde{\alpha}$  is performed independently using Definition 2.2.5, i.e.,

$$(\tilde{3}_{\tilde{\alpha}} \cup \tilde{6}_{\tilde{\alpha}})(x) = [\underline{3}_{\tilde{\alpha}}(x) \vee \underline{6}_{\tilde{\alpha}}(x), \bar{3}_{\tilde{\alpha}}(x) \vee \bar{6}_{\tilde{\alpha}}(x)]$$

$$(\tilde{3}_{\tilde{\alpha}} \cap \tilde{6}_{\tilde{\alpha}})(x) = [\underline{3}_{\tilde{\alpha}}(x) \wedge \underline{6}_{\tilde{\alpha}}(x), \bar{3}_{\tilde{\alpha}}(x) \wedge \bar{6}_{\tilde{\alpha}}(x)]$$

For demonstration the union and intersection of the  $\alpha$ -planes at level  $\tilde{0.2}$  is shown in Table 3.15, then the same task is repeated for all the  $\alpha$ -planes.

Table 3.11. T2FS  $\tilde{3}$ , in Example 3.3.1. The numbers in between are the SGs,  $\tilde{3}_x(u_x)$ .

$x/u_x$	<b>0.0</b>	<b>0.1</b>	<b>0.2</b>	<b>0.3</b>	<b>0.4</b>	<b>0.5</b>	<b>0.6</b>	<b>0.7</b>	<b>0.8</b>	<b>0.9</b>	<b>1.0</b>
<b>1</b>	1.0	0.6	0.3								
<b>2</b>				0.1	0.6	1.0	0.7	0.2			
<b>3</b>											1.0
<b>4</b>				0.1	0.6	1.0	0.7	0.2			
<b>5</b>	1.0	0.6	0.3								

Table 3.12. T2FS  $\tilde{6}$ , in Example 3.3.1. The numbers in between are the SGs,  $\tilde{6}_x(u_x)$ .

$x/u_x$	<b>0.0</b>	<b>0.1</b>	<b>0.2</b>	<b>0.3</b>	<b>0.4</b>	<b>0.5</b>	<b>0.6</b>	<b>0.7</b>	<b>0.8</b>	<b>0.9</b>	<b>1.0</b>
<b>4</b>	1.0	0.8	0.4	0.2	0.1						
<b>5</b>						0.2	1.0	0.4			
<b>6</b>											1.0
<b>7</b>						0.2	1.0	0.4			
<b>8</b>	1.0	0.8	0.4	0.2	0.1						

Table 3.13. The  $\alpha$ -planes,  $\tilde{3}_{\alpha}$ , in Example 3.2.1.

$x$	1	2	3	4	5
$\tilde{3}_{0.0}(x)$	[0, 0.2]	[0.3, 0.7]	[1, 1]	[0.3, 0.7]	[0, 0.2]
$\tilde{3}_{0.1}(x)$	[0, 0.2]	[0.3, 0.7]	[1, 1]	[0.3, 0.7]	[0, 0.2]
$\tilde{3}_{0.2}(x)$	[0, 0.2]	[0.4, 0.7]	[1, 1]	[0.4, 0.7]	[0, 0.2]
$\tilde{3}_{0.3}(x)$	[0, 0.2]	[0.4, 0.6]	[1, 1]	[0.4, 0.6]	[0, 0.2]
$\tilde{3}_{0.4}(x)$	[0, 0.1]	[0.4, 0.6]	[1, 1]	[0.4, 0.6]	[0, 0.1]
$\tilde{3}_{0.5}(x)$	[0, 0.1]	[0.4, 0.6]	[1, 1]	[0.4, 0.6]	[0, 0.1]
$\tilde{3}_{0.6}(x)$	[0, 0.1]	[0.4, 0.6]	[1, 1]	[0.4, 0.6]	[0, 0.1]
$\tilde{3}_{0.7}(x)$	[0, 0]	[0.5, 0.6]	[1, 1]	[0.5, 0.6]	[0, 0]
$\tilde{3}_{0.8}(x)$	[0, 0]	[0.5, 0.5]	[1, 1]	[0.5, 0.5]	[0, 0]
$\tilde{3}_{0.9}(x)$	[0, 0]	[0.5, 0.5]	[1, 1]	[0.5, 0.5]	[0, 0]
$\tilde{3}_1(x)$	[0, 0]	[0.5, 0.5]	[1, 1]	[0.5, 0.5]	[0, 0]

This method requires that the  $\alpha$ -planes of a T2FS to be IVFSs. This assumption allows the use of methods already defined for IVFSs (or IT2FSs) with each  $\alpha$ -plane and thus extended to T2FSs. One of the main advantages of this method is the ability to define operations independently for each  $\alpha$ -plane. This fact suggests the use of parallel or distributed techniques to process operations would highly be successful as discussed in details in Chapter 6 of this thesis. In the following section the  $\alpha$ -plane EP is used to define  $\alpha$ -cuts for T2FSs. The idea is to make use of the  $\alpha$ -cut RT for IVFSs and decompose each  $\alpha$ -plane into  $\alpha$ -cuts. This is key result in the field of T2FSs and systems.

\*In these tabular views of discrete T2FSs the empty cells represent 0.0 grades.

Table 3.14. The  $\alpha$ -planes,  $\tilde{6}_{\tilde{\alpha}}$ , in Example 3.2.1.

$x$	4	5	6	7	8
$\tilde{6}_{\tilde{0}}(x)$	[0, 0.4]	[0.5, 0.7]	[1, 1]	[0.5, 0.7]	[0, 0.4]
$\tilde{6}_{\tilde{0.1}}(x)$	[0, 0.4]	[0.5, 0.7]	[1, 1]	[0.5, 0.7]	[0, 0.4]
$\tilde{6}_{\tilde{0.2}}(x)$	[0, 0.3]	[0.5, 0.7]	[1, 1]	[0.5, 0.7]	[0, 0.3]
$\tilde{6}_{\tilde{0.3}}(x)$	[0, 0.2]	[0.6, 0.7]	[1, 1]	[0.6, 0.7]	[0, 0.2]
$\tilde{6}_{\tilde{0.4}}(x)$	[0, 0.2]	[0.6, 0.7]	[1, 1]	[0.6, 0.7]	[0, 0.2]
$\tilde{6}_{\tilde{0.5}}(x)$	[0, 0.1]	[0.6, 0.6]	[1, 1]	[0.6, 0.6]	[0, 0.1]
$\tilde{6}_{\tilde{0.6}}(x)$	[0, 0.1]	[0.6, 0.6]	[1, 1]	[0.6, 0.6]	[0, 0.1]
$\tilde{6}_{\tilde{0.7}}(x)$	[0, 0.1]	[0.6, 0.6]	[1, 1]	[0.6, 0.6]	[0, 0.1]
$\tilde{6}_{\tilde{0.8}}(x)$	[0, 0.1]	[0.6, 0.6]	[1, 1]	[0.6, 0.6]	[0, 0.1]
$\tilde{6}_{\tilde{0.9}}(x)$	[0, 0]	[0.6, 0.6]	[1, 1]	[0.6, 0.6]	[0, 0]
$\tilde{6}_{\tilde{1}}(x)$	[0, 0]	[0.6, 0.6]	[1, 1]	[0.6, 0.6]	[0, 0]

Table 3.15. The union of  $\alpha$ -planes,  $\tilde{3}_{\tilde{0.2}} \cup \tilde{6}_{\tilde{0.2}}$ , in Example 3.2.1.

$x$	$\tilde{3}_{\tilde{0.2}}(x) \wedge \tilde{6}_{\tilde{0.2}}(x)$	$\tilde{3}_{\tilde{0.2}}(x) \vee \tilde{6}_{\tilde{0.2}}(x)$
1	[0, 0]	[0, 0.2]
2	[0, 0]	[0.4, 0.7]
3	[0, 0]	[1, 1]
4	[0, 0.3]	[0.4, 0.7]
5	[0, 0.2]	[0.5, 0.7]
6	[0, 0]	[1, 1]
7	[0, 0]	[0.5, 0.7]
8	[0, 0]	[0, 0.3]

### 3.3 Alpha-cuts of Type-2 Fuzzy Sets

In the previous section  $\alpha$ -cuts for IVFSs are discussed. These  $\alpha$ -cuts can be defined in different ways, what is important though, is that these  $\alpha$ -cuts are defined by crisp sets and the IVFS  $\alpha$ -EP extends operations directly from crisp sets to IVFSs. This fact is crucial since in Section 3.2  $\alpha$ -planes are shown to be IVFSs, and the  $\alpha$ -plane EP is defined in order to enable the extension of operations from IVFSs to T2FSs. Combining these two theorems leads to the definition of  $\alpha$ -cuts for T2FSs. First, the UMF and LMF of  $\alpha$ -planes are generalised. Let,  $\tilde{A} \in \tilde{F}(X)$ , be a T2FS and,  $\tilde{A}_{\tilde{\alpha}} \in \tilde{F}(X)$ , be a IVFS representing its  $\alpha$ -plane at level  $\tilde{\alpha}$ , such that  $\tilde{A}_{\tilde{\alpha}}(x) = [\underline{u}_{x,\tilde{\alpha}}, \bar{u}_{x,\tilde{\alpha}}]$ . Let,  $\underline{A}_{\tilde{\alpha}} \in F(X)$ , be the LMF of  $\tilde{A}_{\tilde{\alpha}}$  and  $\bar{A}_{\tilde{\alpha}} \in F(X)$ , be the UMF of  $\tilde{A}_{\tilde{\alpha}}$ . Then each  $\alpha$ -plane is completely determined by its LMF and UMF, i.e.,

$$\tilde{A}_{\tilde{\alpha}} = (\underline{A}_{\tilde{\alpha}}, \bar{A}_{\tilde{\alpha}}) \quad (3.19)$$

where  $\tilde{A}_{\tilde{\alpha}}(x) = [\underline{A}_{\tilde{\alpha}}(x), \bar{A}_{\tilde{\alpha}}(x)]$ ,  $\underline{A}_{\tilde{\alpha}}(x) = \underline{u}_{x,\tilde{\alpha}}$  and  $\bar{A}_{\tilde{\alpha}}(x) = \bar{u}_{x,\tilde{\alpha}}$ . It is clear that both the LMF and UMF are FSs. Now, taking the  $\alpha$ -cuts of each  $\alpha$ -plane leads to the following definition:

**Definition 3.3.1 (T2  $\alpha$ -cuts)** Let,  $\tilde{A} \in \tilde{F}(X)$ , be a T2FS and,  $\tilde{A}_{\tilde{\alpha}} = (\underline{A}_{\tilde{\alpha}}, \overline{A}_{\tilde{\alpha}})$ , be its  $\alpha$ -plane at level  $\tilde{\alpha}$  represented by its LMF and UMF. Then,  $\tilde{A}_{\tilde{\alpha}, \alpha}$ , is the  $\alpha$ -cut of that  $\alpha$ -plane at level  $\alpha$ , i.e.,

$$\tilde{A}_{\tilde{\alpha}, \alpha} = (\underline{A}_{\tilde{\alpha}, \alpha}, \overline{A}_{\tilde{\alpha}, \alpha}) \quad (3.20)$$

where  $\underline{A}_{\tilde{\alpha}, \alpha}$  and  $\overline{A}_{\tilde{\alpha}, \alpha}$  are the  $\alpha$ -cuts of the LMF and UMF of  $\alpha$ -plane,  $\tilde{A}_{\tilde{\alpha}}$ , respectively.

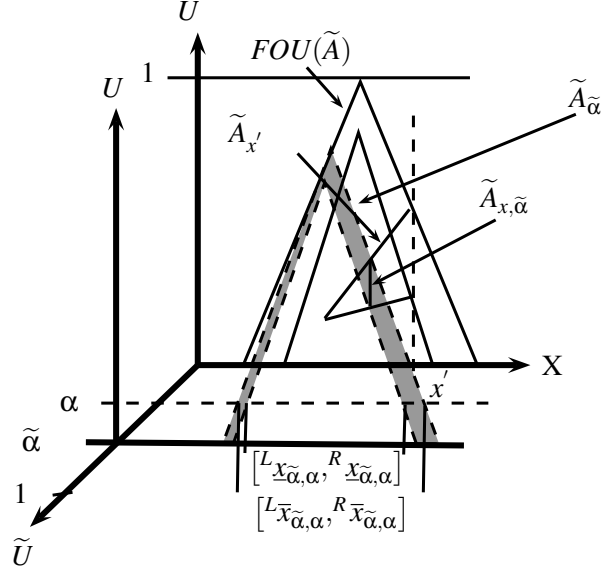


Fig. 3.4. The  $\alpha$ -cut  $\tilde{A}_{\tilde{\alpha}, \alpha}$  of  $\alpha$ -plane  $\tilde{A}_{\tilde{\alpha}}$  of T2FS  $\tilde{A}$ .

Figure 3.4 shows the  $\alpha$ -cut of a continuous T2FS. The  $\alpha$ -cuts of LMF and UMF are crisp sets since the LMF and UMF are FSs. Hence,  $\underline{A}_{\tilde{\alpha}, \alpha}(x) \in \{0, 1\}$ , and  $\overline{A}_{\tilde{\alpha}, \alpha}(x) \in \{0, 1\}$ . Following Definition 3.1.2 the  $\alpha$ -IVFS of each  $\alpha$ -cut is defined, i.e.,

**Definition 3.3.2** For each  $\alpha$ -cut,  $\tilde{A}_{\tilde{\alpha}, \alpha}$ , of the T2FS,  $\tilde{A}$ , a special IVFS ( $\alpha$ -IVFS),  $\alpha\tilde{A}_{\tilde{\alpha}, \alpha} \in \hat{F}(X)$ , can be defined as follows:

$$\begin{aligned} \alpha\tilde{A}_{\tilde{\alpha}, \alpha} &= \alpha(\underline{A}_{\tilde{\alpha}, \alpha}, \overline{A}_{\tilde{\alpha}, \alpha}) \\ &= (\alpha\underline{A}_{\tilde{\alpha}, \alpha}, \alpha\overline{A}_{\tilde{\alpha}, \alpha}) \end{aligned} \quad (3.21)$$

where  $\alpha\tilde{A}_{\tilde{\alpha}, \alpha}(x) = \alpha \wedge [\underline{A}_{\tilde{\alpha}, \alpha}(x), \overline{A}_{\tilde{\alpha}, \alpha}(x)] = [\alpha \wedge \underline{A}_{\tilde{\alpha}, \alpha}(x), \alpha \wedge \overline{A}_{\tilde{\alpha}, \alpha}(x)]$ .

It is noticeable that  $\alpha A_{\tilde{\alpha},\alpha}$  and  $\alpha \bar{A}_{\tilde{\alpha},\alpha}$  are  $\alpha$ -FSs. The union of all  $\alpha$ -IVFSs constitute an  $\alpha$ -plane.

$$\begin{aligned}\tilde{A}_{\tilde{\alpha}} &= \bigcup_{\forall \alpha} \alpha \tilde{A}_{\tilde{\alpha},\alpha} \\ &= \bigcup_{\forall \alpha} \alpha (\underline{A}_{\tilde{\alpha},\alpha}, \bar{A}_{\tilde{\alpha},\alpha})\end{aligned}\quad (3.22)$$

Earlier in Eq. 2.64 a special T2FS called  $\alpha$ -T2FS which is associated with each  $\alpha$ -plane was defined. Substituting the  $\alpha$ -plane  $\tilde{A}_{\tilde{\alpha}}$  from Eq. 3.22 in Eq. 2.64 gives the following  $\alpha$ -T2FS:

$$\begin{aligned}\tilde{\alpha} \tilde{A}_{\tilde{\alpha}} &= \tilde{\alpha} \bigcup_{\forall \alpha} \alpha \tilde{A}_{\tilde{\alpha},\alpha} \\ &= \tilde{\alpha} \bigcup_{\forall \alpha} \alpha (\underline{A}_{\tilde{\alpha},\alpha}, \bar{A}_{\tilde{\alpha},\alpha})\end{aligned}\quad (3.23)$$

where

$$\left( \tilde{\alpha} \bigcup_{\forall \alpha} \alpha \tilde{A}_{\tilde{\alpha},\alpha}(x) \right) (u_{x,\tilde{\alpha}}) = \tilde{\alpha} \wedge \left( \bigcup_{\forall \alpha} \alpha \tilde{A}_{\tilde{\alpha},\alpha}(x) \right) (u_{x,\tilde{\alpha}})$$

and

$$\bigcup_{\forall \alpha} \alpha \tilde{A}_{\tilde{\alpha},\alpha}(x)(u_{x,\tilde{\alpha}}) = \begin{cases} 1, & u_{x,\tilde{\alpha}} \in \bigcup_{\forall \alpha} \alpha \tilde{A}_{\tilde{\alpha},\alpha}(x) \\ 0, & u_{x,\tilde{\alpha}} \notin \bigcup_{\forall \alpha} \alpha \tilde{A}_{\tilde{\alpha},\alpha}(x) \end{cases}$$

It is already known from the  $\alpha$ -plane RT that a T2FS can be represented by the union of all such  $\alpha$ -T2FSs. Basically, these  $\alpha$ -T2FSs are associated T2FSs which associates a given  $\alpha$ -plane with level  $\tilde{\alpha}$  on  $\tilde{U}$ . Each  $\alpha$ -plane is an IVFS which can be represented by the union of all  $\alpha$ -IVFSs, which are associated IVFS that associate a given  $\alpha$ -cut with level  $\alpha$  on  $U$ . Since the  $\alpha$ -IVFSs are completely determined by their UMF and LMF, the  $\alpha$ -cuts of the LMF and UMF are used instead. These  $\alpha$ -cuts are crisp sets, and hence the overall picture is clear, and the original T2FS can be represented using the union of all such sets. The following representation theorem is by far the most significant result of this thesis. It is mathematically simple, semantically sound, and as shown in the following three chapters theoretically and practically important.

**Theorem 3.3.1 (T2FS  $\alpha$ -cut RT)** *A T2FS,  $\tilde{A}$ , can be represented by the union of all its  $\alpha$ -T2FSs, i.e.,*

$$\tilde{A} = \bigcup_{\forall \tilde{\alpha}} \tilde{\alpha} \bigcup_{\forall \alpha} \alpha \tilde{A}_{\tilde{\alpha},\alpha} \quad (3.24)$$

**Proof.** Straight forward substitute Eq. 3.23 in Eq. 2.65 of Theorem 2.8.1. ■

The  $\alpha$ -cut representation allows T2FSs to be decomposed into its smallest interpretable components, i.e., crisp sets while maintaining the relationship between domain values by their degree of membership. T2FSs can be looked upon as weighted crisp sets with the PGs and SGs as weighting

factors. In fact, T2FSs can be represented by equivalent FSs. It also suggests that it is the same for higher types- of fuzzy sets such as type-n fuzzy sets. The VS,  $\alpha$ -plane and  $\alpha$ -cut representations are by definition related. The relationship between these representations is depicted in Figure 3.5.

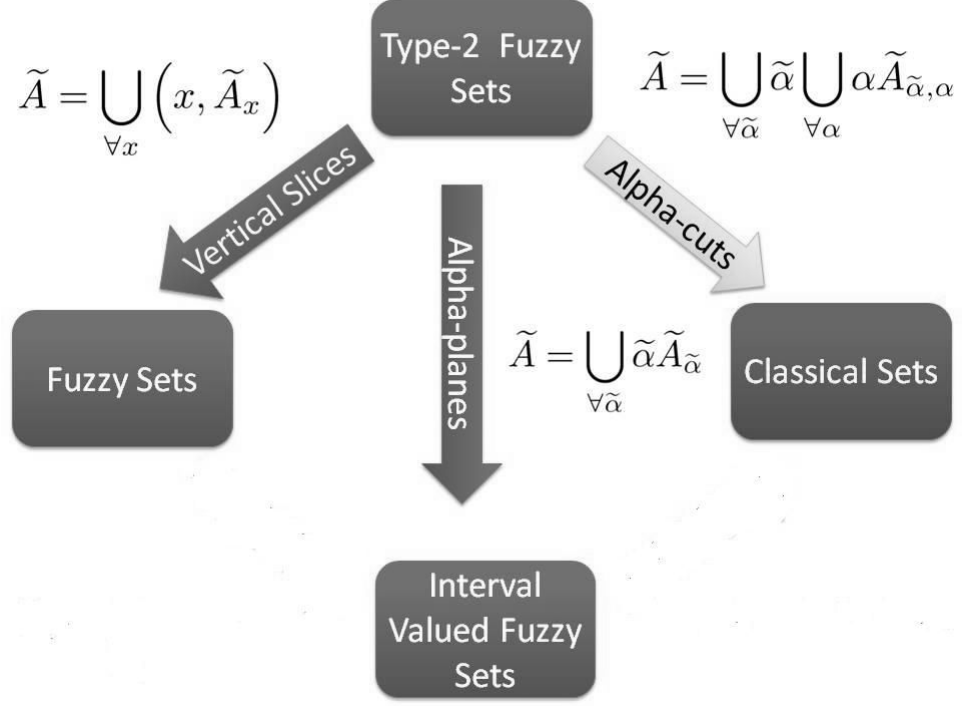


Fig. 3.5. The vertical slice,  $\alpha$ -plane and  $\alpha$ -cut representations of T2FSs and their relationship.

The relation between domain values in the classical set theoretic manner is behind the idea of  $\alpha$ -cuts for FSs. This relation is maintained across IVFSs and T2FSs as they are extension of classical FSs. Figure 3.6 shows this relationship, the  $\alpha$ -cut RT of fuzzy sets decomposes these sets into classical sets, the  $\alpha$ -plane RT decomposes T2FSs into IVFSs, and the  $\alpha$ -cut RT of IVFSs decomposes them into classical sets. Although the  $\alpha$ -cut decomposition of T2FSs is the most significant result of this thesis, what makes such decomposition even more interesting is the ability to perform operations on T2FSs in the classical set theoretic sense. A capability similar to that of the  $\alpha$ -cut EP for FSs, and that of the IVFS  $\alpha$ -cut EP presented in Theorem 3.1.2. This is made possible by extending the  $\alpha$ -EP of FSs to IVFSs, and by the  $\alpha$ -plane EP of T2FSs. If not for the representation theorem that made the following extension principle possible, the following theorem would have been the most important result of this thesis.

**Theorem 3.3.2 (T2FS  $\alpha$ -cut EP)** *Let,  $X = X_1 \times \dots \times X_n$ , be the Cartesian product of universes, and  $\tilde{A}_1, \dots, \tilde{A}_n$  be T2FSs in each universe respectively. Also let  $Y$  be another universe and  $\tilde{B} \in Y$*



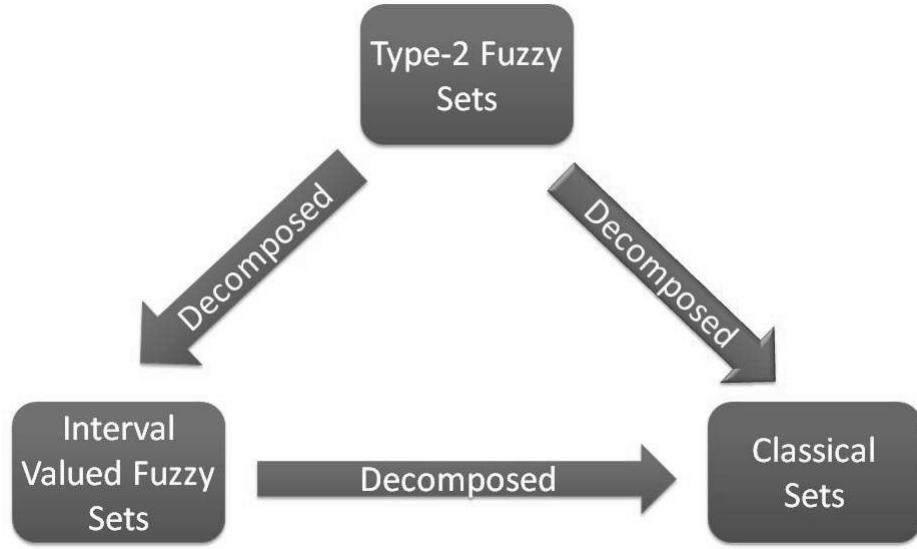


Fig. 3.6. The relationship between T2FSs, IVFSs and crisp sets.

be a T2FS such that  $\tilde{B} = f(\tilde{A}_1, \dots, \tilde{A}_n)$ , where  $f : X \rightarrow Y$  is a monotone mapping. Then  $\tilde{B}$  is equal to the union of applying the same function to all its decomposed  $\alpha$ -cuts, i.e.,

$$\begin{aligned} \tilde{B} &= f(\tilde{A}_1, \dots, \tilde{A}_n) \\ &= \bigcup_{\forall \tilde{\alpha}} \tilde{\alpha} \bigcup_{\forall \alpha} \alpha f(\tilde{A}_{1_{\tilde{\alpha}, \alpha}}, \dots, \tilde{A}_{n_{\tilde{\alpha}, \alpha}}) \end{aligned}$$

**Proof.** From Theorem 3.2.1 operations are extended to T2FSs by the  $\alpha$ -plane EP from operations on its  $\alpha$ -planes which are IVFSs. For each  $\alpha$ -plane Theorem 3.1.2 allows the operations to be extended from crisp sets to IVFSs. Hence, straight forward substitute Eq. 3.7 in Eq. 3.11 and that completes the proof. ■

This novel theorem bares the possibility of being a fundamental theorem in the field of fuzzy sets and indeed could be used extensively by the type-2 fuzzy logic community. This fundamental theorem has the potential to serve as a great tool in covering the gap between uncertainty theories and T2FSs. The following example demonstrates how to use Theorem 3.3.2 for defining operations for T2FSs by calculating the join and meet of a T2FS using the  $\alpha$ -cut extension principle.

**Example 3.3.1** Consider the T2FSs,  $\tilde{3}$ , in Table 3.11 and,  $\tilde{6}$ , in Table 3.12. To perform the join, a decomposition of each T2FS into its  $\alpha$ -planes and each  $\alpha$ -plane to its  $\alpha$ -cuts must be performed. Then, for example the union of  $\alpha$ -planes  $\tilde{3}_{0.2} \cup \tilde{6}_{0.2}$ , is computed. The interval membership grades of each  $\alpha$ -plane are constructed using the bounds of the PMs  $J_{x, \tilde{\alpha}}$ , i.e. Table 3.16 and Table 3.17.

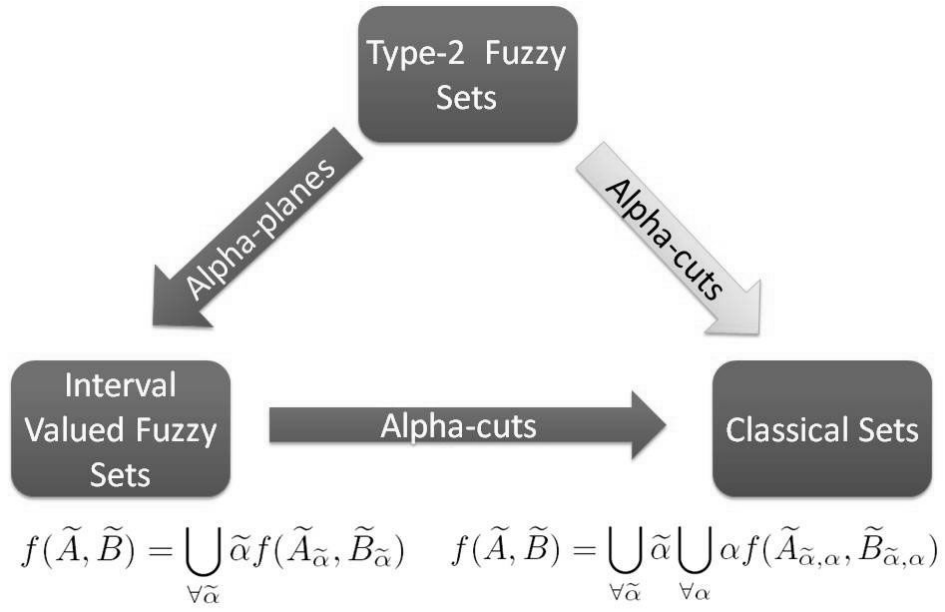


Fig. 3.7. The  $\alpha$ -plane EP and the  $\alpha$ -cut EP T2FSs.

The steps to perform the union is shown in Table 3.18, Table 3.19 and Table 3.20. These are the same steps used to perform the union of IVFSs. To perform the union of the T2FSs the same task is repeated for all the  $\alpha$ -planes.

Table 3.16.  $\alpha$ -plane,  $\hat{3}_{0.2}$ , in Example 3.3.1.

$x$	1	2	3	4	5
$\hat{3}_{0.2}(x)$	[0, 0.2]	[0.4, 0.7]	[1, 1]	[0.4, 0.7]	[0, 0.2]

Table 3.17.  $\alpha$ -plane,  $\tilde{6}_{0.2}$ , in Example 3.3.1.

$x$	4	5	6	7	8
$\tilde{6}_{0.2}(x)$	[0, 0.3]	[0.5, 0.7]	[1, 1]	[0.5, 0.7]	[0, 0.3]

Table 3.18. The  $\alpha$ -cuts of  $\alpha$ -planes,  $\tilde{3}_{0.2}$  and  $\tilde{6}_{0.2}$ , in Example 3.3.1.

$\alpha$	$\tilde{3}_{0.2,\alpha}$	$\tilde{6}_{0.2,\alpha}$	$\tilde{3}_{0.2,\alpha}$	$\tilde{6}_{0.2,\alpha}$
0.0	{1, 2, 3, 4, 5}	{4, 5, 6, 7, 8}	{1, 2, 3, 4, 5}	{4, 5, 6, 7, 8}
0.1	{2, 3, 4}	{5, 6, 7}	{1, 2, 3, 4, 5}	{4, 5, 6, 7, 8}
0.2	{2, 3, 4}	{5, 6, 7}	{1, 2, 3, 4, 5}	{4, 5, 6, 7, 8}
0.3	{2, 3, 4}	{5, 6, 7}	{2, 3, 4}	{4, 5, 6, 7, 8}
0.4	{2, 3, 4}	{5, 6, 7}	{2, 3, 4}	{5, 6, 7}
0.5	{3}	{5, 6, 7}	{2, 3, 4}	{5, 6, 7}
0.6	{3}	{6}	{2, 3, 4}	{5, 6, 7}
0.7	{3}	{6}	{2, 3, 4}	{5, 6, 7}
0.8	{3}	{6}	{3}	{6}
0.9	{3}	{6}	{3}	{6}
1.0	{3}	{6}	{3}	{6}

Table 3.19. The  $\alpha$ -cuts of  $\alpha$ -planes,  $\tilde{3}_{0.2} \cup \tilde{6}_{0.2}$ , in Example 3.3.1.

$\alpha$	$\tilde{3}_{0.2,\alpha} \cup \tilde{6}_{0.2,\alpha}$	$\tilde{3}_{0.2,\alpha} \cup \tilde{6}_{0.2,\alpha}$
0.0	{1, 2, 3, 4, 5, 6, 7, 8}	{1, 2, 3, 4, 5, 6, 7, 8}
0.1	{2, 3, 4, 5, 6, 7}	{1, 2, 3, 4, 5, 6, 7, 8}
0.2	{2, 3, 4, 5, 6, 7}	{1, 2, 3, 4, 5, 6, 7, 8}
0.3	{2, 3, 4, 5, 6, 7}	{2, 3, 4, 5, 6, 7, 8}
0.4	{2, 3, 4, 5, 6, 7}	{2, 3, 4, 5, 6, 7}
0.5	{3, 5, 6, 7}	{2, 3, 4, 5, 6, 7}
0.6	{3, 6}	{2, 3, 4, 5, 6, 7}
0.7	{3, 6}	{2, 3, 4, 5, 6, 7}
0.8	{3, 6}	{3, 6}
0.9	{3, 6}	{3, 6}
1.0	{3, 6}	{3, 6}

This theorem has been put forward for the first time in (Hamrawi & Coupland 2009b). In this section  $\alpha$ -cuts for T2FSs and its associated T2FS  $\alpha$ -EP are defined. There are now available different ways to extend operations to FSs and its extensions. Specifically in this chapter two of these extension principle are defined for T2FSs, the  $\alpha$ -plane EP and the T2FS  $\alpha$ -cut EP. The relationship between these two extension principles are shown in Figure 3.7. While the  $\alpha$ -plane EP extends functions and operations from IVFSs to T2FSs directly, the T2FS  $\alpha$ -cut EP extends functions and operations from crisp sets directly, both from a set-wise perspective. The  $\alpha$ -cut RT definitely serves great interest in practical and theoretical investigations of T2FSs. The rest of the thesis is dedicated to highlight the effectiveness of the methods defined in this chapter, both theoretically and practically. Theoretically in Chapters 4 and 5 uncertainty measures and type-2 fuzzy numbers along with arithmetic operations are defined in a straight forward manner. Practically, in Chapter 6 it is shown that by definition the  $\alpha$ -cut RT is suitable for massively parallel processing, and some functions and operation on T2FSs are performed on a massively GPU computing device. Towards the end of the pinnacle section of this thesis, it is worth highlighting that the  $\alpha$ -cut RT shown in

Table 3.20.  $\alpha$ -plane,  $\tilde{3}_{0.2,\alpha} \cup \tilde{6}_{0.2,\alpha}$ , in Example 3.3.1 from its  $\alpha$ -cuts in Table 3.19.

$x$	$\alpha(\tilde{3}_{0.2,\alpha} \cup \tilde{6}_{0.2,\alpha})(x)$	$\alpha(\tilde{3}_{0.2,\alpha} \cup \tilde{6}_{0.2,\alpha})(x)$	$(\tilde{3}_{0.2,\alpha} \cup \tilde{6}_{0.2,\alpha})(x)$
1	0	0, 0.1, 0.2	[0, 0.2]
2	0, 0.1, 0.2, 0.3, 0.4	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7	[0.4, 0.7]
3	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1	[1, 1]
4	0, 0.1, 0.2, 0.3, 0.4	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7	[0.4, 0.7]
5	0, 0.1, 0.2, 0.3, 0.4, 0.5	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7	[0.5, 0.7]
6	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1	[1, 1]
7	0, 0.1, 0.2, 0.3, 0.4, 0.5	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7	[0.5, 0.7]
8	0	0, 0.1, 0.2, 0.3	[0, 0.3]

Theorem 3.3.1, and the  $\alpha$ -cut EP shown in Theorem 3.3.2 are the main results of this thesis.

### 3.4 More Definitions for T2FSs

Earlier in Section 2.1, some important notions have been defined. These notions are used extensively in the literature of fuzzy sets in order to communicate about different aspects of fuzzy sets. These notions include the height, support, and core of fuzzy sets. Moreover, some notions describe different situations in which fuzzy sets can be categorised and classified within. These notions include normality, convexity and containment. In this section these notions are extended to IVFSs and T2FSs. In addition to these notions some properties and variations of the  $\alpha$ -plane and  $\alpha$ -cut representations are discussed for the first time. In order to begin this investigation, an essential rule called the *reduction rule* is introduced in the following definition.

**Definition 3.4.1** When all uncertainties<sup>†</sup> about the membership grades of a T2FS disappear, a T2FS reduces to a FS.

Figure 3.8 depicts this rule for a T2FS with triangular PMs and trapezoidal VSs. This rule can actually be applied in different stages. When the uncertainties about the secondary membership function disappears a T2FS actually reduces to an IVFS. When uncertainty about the interval membership grades disappear an IVFS reduces to a FS. Finally, if the uncertainty disappears completely a FS reduces to a crisp set. This reduction pattern should be intact throughout any operation performed or notion defined for T2FSs. Failure to satisfy this rule may be considered as a deficiency for some operations specially if these operations are required in comparative studies or for any practical reason. For example Mendel (2007) considered the operations defined using the geometric representation of T2FSs, incapable of reducing to FSs when uncertainty disappear. So this reduction rule is in fact stated implicitly by Mendel (2007), and Mendel (2001) also stated a similar statement describing the PMF. In the definition above this rule is made more significant by

<sup>†</sup>Uncertainties about the membership grades of a T2FS are represented by a fuzzy sets.

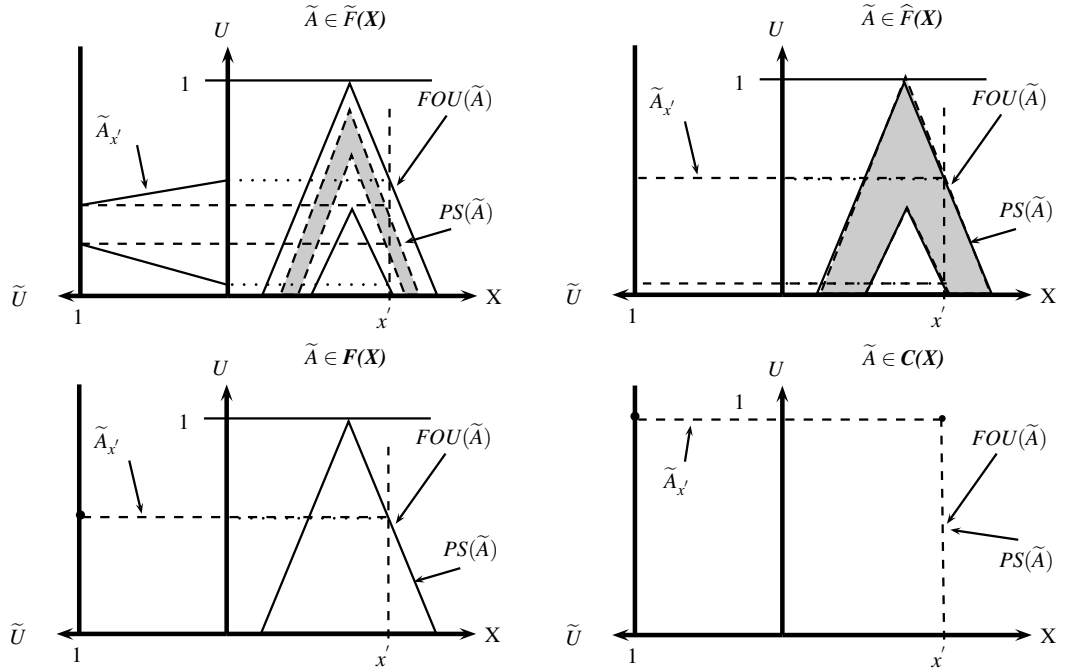


Fig. 3.8. Explanation of the reduction rule.

formalising it into a definition. It also formulates the basis for most of the definitions introduced in this section. To explain this rule, consider a T2FS  $\tilde{A} \in \tilde{F}(X)$  where,

$$\tilde{A} = \left\{ ((x, u_x), \tilde{u}_x) \mid x \in X, u_x \in J_x \subseteq U, \tilde{u}_x = \tilde{A}(x, u_x) \in \tilde{U} \right\} \quad (3.25)$$

Here for each  $x$ , its membership grade is a FS, i.e.,

$$\tilde{A}(x) = \{(u_x, \tilde{u}_x) \mid \forall u_x \in J_x\}$$

Usually but not always,  $J_x$  is considered to be an interval either by definition or through a bounding operation. For this T2FS to become an IT2FS, then the secondary grades should all be at unity, i.e.  $\forall \tilde{u}_x = 1$  which is actually equal to an IVFS.

$$\tilde{A} = \{((x, u_x), 1) \mid x \in X, u_x \in J_x \subseteq U\} \quad (3.26)$$

Since  $J_x$  is an interval then for each  $x$ , its membership grade can be defined by an interval  $\tilde{A}(x) = [\underline{u}_x, \bar{u}_x]$ . This situation describes the disappearance of uncertainties from the secondary membership function or third dimension. If the uncertainty about the primary grades disappear in such a

way that they become values in  $U$  rather than intervals.

$$\tilde{A} = \{((x, u_x), 1) \mid x \in X, u_x \in U\} \quad (3.27)$$

Here,  $\tilde{A}$  represents a fuzzy set, then for each  $x$ , its membership grade  $\tilde{A}(x) = u_x$ . When all uncertainty disappear the T2FS becomes a crisp set as follows:

$$\tilde{A} = \{((x, 1), 1)\} \quad (3.28)$$

Table 3.21. The reduction rule using point-valued and set-valued definitions.

Set	point-valued	set-valued
T2FS	$\tilde{A} = \{((x, u_x), \tilde{u}_x)\}$ where $x \in X, u_x \in J_x \subseteq U$ , and $\tilde{u}_x = \tilde{A}(x, u_x) \in \tilde{U}$	$\bigcup_{\forall \tilde{\alpha}} \tilde{\alpha} \tilde{A}_{\tilde{\alpha}}$ $= \bigcup_{\forall \tilde{\alpha}} \tilde{\alpha} (\underline{A}_{\tilde{\alpha}}, \bar{A}_{\tilde{\alpha}})$ $= \bigcup_{\forall \tilde{\alpha}} \tilde{\alpha} \bigcup_{\forall \alpha} \alpha (\underline{A}_{\tilde{\alpha}, \alpha}, \bar{A}_{\tilde{\alpha}, \alpha})$
IVFS	$\tilde{A} = \{((x, u_x), 1)\}$ where $x \in X, u_x \in J_x \subseteq U$	$\bigcup_{\forall \tilde{\alpha}} \tilde{\alpha} \tilde{A}_{\tilde{\alpha}}$ where $\forall \tilde{\alpha}_i, \tilde{\alpha}_j \in \tilde{U}, \tilde{\alpha}_i \neq \tilde{\alpha}_j, \tilde{A}_{\tilde{\alpha}_i} = \tilde{A}_{\tilde{\alpha}_j}$ , and $\tilde{A}_{\tilde{\alpha}=1} \neq \emptyset$
FS	$\tilde{A} = \{((x, u_x), 1)\}$ where $x \in X, u_x \in U$	$\bigcup_{\forall \tilde{\alpha}} \tilde{\alpha} (\underline{A}_{\tilde{\alpha}}, \bar{A}_{\tilde{\alpha}})$ where $\forall \tilde{\alpha}_i, \tilde{\alpha}_j \in \tilde{U}, \tilde{\alpha}_i \neq \tilde{\alpha}_j, \underline{A}_{\tilde{\alpha}_i} = \bar{A}_{\tilde{\alpha}_i} = \underline{A}_{\tilde{\alpha}_j} = \bar{A}_{\tilde{\alpha}_j}$ , and $\underline{A}_{\tilde{\alpha}=1} = \bar{A}_{\tilde{\alpha}=1} \neq \emptyset$
Crisp	$\tilde{A} = \{((x, 1), 1)\}$ where $x \in X$	$= \bigcup_{\forall \tilde{\alpha}} \tilde{\alpha} \bigcup_{\forall \alpha} \alpha (\underline{A}_{\tilde{\alpha}, \alpha}, \bar{A}_{\tilde{\alpha}, \alpha})$ where $\forall \tilde{\alpha}_i, \tilde{\alpha}_j \in \tilde{U}, \tilde{\alpha}_i \neq \tilde{\alpha}_j$ , $\forall \alpha_k, \alpha_l \in U, \alpha_k \neq \alpha_l$ , $\underline{A}_{\tilde{\alpha}_i, \alpha_k} = \bar{A}_{\tilde{\alpha}_i, \alpha_k} = \underline{A}_{\tilde{\alpha}_j, \alpha_l} = \bar{A}_{\tilde{\alpha}_j, \alpha_l}$ , and $\underline{A}_{\tilde{\alpha}=1, \alpha=1} = \bar{A}_{\tilde{\alpha}=1, \alpha=1} \neq \emptyset$

It is clear that  $\tilde{A}(x) \in \{0, 1\}$ . Any function to be introduced for T2FSs should at least satisfy the reduction rule of Definition 3.4.1. Definitions of fuzzy sets can be defined using both point-valued and set-valued forms, making use of the  $\alpha$ -cut RT for the latter. In the same way different definitions for T2FSs are defined utilising the  $\alpha$ -plane and  $\alpha$ -cut representations in order to provide set-valued forms. Table 3.21 shows this fact, in the first row both forms are represented and it is clear that set-valued form contains the  $\alpha$ -plane representation with its two variations and the  $\alpha$ -cut

representation. In the second row the  $\alpha$ -plane RT is used to define an IVFS.

$$\tilde{A} = \bigcup_{\forall \tilde{\alpha}} \tilde{\alpha} \tilde{A}_{\tilde{\alpha}} \mid \forall \tilde{\alpha}_i, \tilde{\alpha}_j \in \tilde{U}, \tilde{\alpha}_i \neq \tilde{\alpha}_j, \tilde{A}_{\tilde{\alpha}_i} = \tilde{A}_{\tilde{\alpha}_j} \text{ and } \tilde{A}_{\tilde{\alpha}=1} \neq \emptyset \quad (3.29)$$

This equation shows the condition when T2FS  $\tilde{A}$  reduces to an IVFS. It is the case when all its  $\alpha$ -planes are equal, and the  $\alpha$ -plane at  $\tilde{\alpha} = 1$  exists. In other words all the  $\alpha$ -planes are equal to the principal set<sup>‡</sup>,  $PS(\tilde{A}) = \tilde{A}_1$ . It is clear that as a consequence the FOU is equal to the PS, i.e.,  $\tilde{A}_0 = \tilde{A}_1$ . In the third row of Table 3.21 the T2FS is reduced to a FS.

$$\tilde{A} = \bigcup_{\forall \tilde{\alpha}} \tilde{\alpha} (\underline{A}_{\tilde{\alpha}}, \overline{A}_{\tilde{\alpha}}) \mid \forall \tilde{\alpha}_i, \tilde{\alpha}_j \in \tilde{U}, \tilde{\alpha}_i \neq \tilde{\alpha}_j, \underline{A}_{\tilde{\alpha}_i} = \overline{A}_{\tilde{\alpha}_i} = \underline{A}_{\tilde{\alpha}_j} = \overline{A}_{\tilde{\alpha}_j} \text{ and } \underline{A}_{\tilde{\alpha}=1} = \overline{A}_{\tilde{\alpha}=1} \neq \emptyset \quad (3.30)$$

This equation shows the condition when the T2FS  $\tilde{A}$  reduces to a FS. It is the case when all the LMFs and UMFs of all its  $\alpha$ -planes are equal, and the PS also exist. In fact the PS itself reduces to be Mendel's principal membership function<sup>§</sup>. In other words all the LMFs and UMFs of all the  $\alpha$ -planes are equal to the PMF,  $PMF(\tilde{A}) = \underline{A}_1 = \overline{A}_1$ . It is also clear that as a consequence the FOU itself is a FS that equals the PMF, i.e.,  $\underline{A}_0 = \overline{A}_0 = PMF(\tilde{A})$ . In the fourth row of Table 3.21 the T2FS is reduced to a crisp set.

$$\begin{aligned} \tilde{A} &= \bigcup_{\forall \tilde{\alpha}} \tilde{\alpha} \bigcup_{\forall \alpha} \alpha (\underline{A}_{\tilde{\alpha}, \alpha}, \overline{A}_{\tilde{\alpha}, \alpha}) \text{ where} \\ &\forall \tilde{\alpha}_i, \tilde{\alpha}_j \in \tilde{U}, \tilde{\alpha}_i \neq \tilde{\alpha}_j, \forall \alpha_k, \alpha_l \in U, \alpha_k \neq \alpha_l, \underline{A}_{\tilde{\alpha}_i, \alpha_k} = \overline{A}_{\tilde{\alpha}_i, \alpha_k} = \underline{A}_{\tilde{\alpha}_j, \alpha_l} = \overline{A}_{\tilde{\alpha}_j, \alpha_l}, \\ &\text{and } \underline{A}_{\tilde{\alpha}=1, \alpha=1} = \overline{A}_{\tilde{\alpha}=1, \alpha=1} \neq \emptyset \end{aligned} \quad (3.31)$$

This equation shows the condition when T2FS  $\tilde{A}$  reduces to a crisp set. It is the case when all its  $\alpha$ -cuts are equal, and the PMF also exist. In fact the PMF itself reduces to a crisp set. To apply this reduction pattern to define the height, support, core, and containment of T2FSs they should at least reduce to their respective definitions in FSs. To this end in the following a progressive approach from FSs to IVFSs to T2FSs is used. The height of fuzzy set is defined to be the highest membership degree attained by any domain value in the given FS. For the FS,  $A \in \mathbf{F}(X)$ , its height is defined in Definition 2.1.7, and for clarity it is shown again.

$$h(A) = \sup_{\forall x} A(x)$$

In IVFSs it is apparent that there are two heights, one represents the height of the LMF and the other the height of the UMF. So a direct extension can be applied to IVFSs from FSs. Let  $\hat{A} \in \hat{\mathbf{F}}(X)$

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<sup>‡</sup>Refer to Definition 2.3.9.

<sup>§</sup>Refer to Definition 2.3.8.

be an IVFS with interval membership grade  $\hat{A}(x) = [\underline{u}_x, \bar{u}_x]$  then the height  $h(\hat{A})$  which represents the highest grade of membership attained by any domain value in the set can be described by the following equation:

$$h(\hat{A}) = \sup_{\forall x} \hat{A}(x) = \sup_{\forall x} [\underline{u}_x, \bar{u}_x] \quad (3.32)$$

This is a direct extension to the FS definition of height. Two problems appear by this definition, first, mathematically in some cases the height of the LMF and the UMF may not occur at the same point of the domain. Second, semantically the information needed when asking about the height of an IVFS, should be a single number. Basically, the information about the height of a FS is not about the membership grade of the FS, but about the domain value which is represented by that height. The question is about the domain value that attains the highest degree of belonging. If this view is adopted, then the LMF does not serve the purpose. By definition the UMF is closer to unity than the LMF, consequently it is more appropriate to associate the height of the IVFS to the height of the UMF.

**Definition 3.4.2** Let  $\hat{A} \in \hat{F}(X)$  then the height  $h(\hat{A})$  is the highest UMF grade of membership attained by any domain value in the set, i.e.,

$$h(\hat{A}) = h(\bar{A}) = \sup_{\forall x} \bar{A}(x)$$

In T2FSs it is well understood that there are three dimensions. To be able to provide a meaningful situation that represents the height of the T2FS, it should give the same semantics as it provides for the FS and IVFS. Here the primary grades and secondary grades play different roles. To make this situation clear, first, let  $\tilde{A} \in \tilde{F}(X)$  be a T2FS with vertical slices  $\tilde{A}_x$  at each domain value  $x$ . Recall that for every PG,  $u_x$ , there exist only one SG,  $\tilde{u}_x$ , associated with it, i.e.  $\tilde{A}_x = \bigcup_{u_x \in U} (u_x, \tilde{u}_x)$ , where  $u_x \in J_x \subseteq U$  and  $\tilde{u}_x \in \tilde{U}$ . Second, recall the fact that a T2FS is reduced to a crisp set when both the PG and SG are at unity, i.e.  $u_x = \tilde{u}_x = 1$ . The third fact is that it is not always the case when the maximum PG is attained for all domain values, the SG also attains its maximum. Also it is not always the case when the maximum SG is attained for all domain values, the PG also attains its maximum. The question in place is the following: which of the two adds more weight to the T2FS, is it the PG or SG? or are they of equal weight? This question can be looked at mathematically by the following example, assume that the maximum PG is 0.6 and its associated SG is  $\widetilde{0.4}$ , let it be written as  $(u_{x_1}, \tilde{u}_{x_1}) = (0.6, \widetilde{0.4})$ . On the other hand assume that the maximum SG is  $\widetilde{0.6}$  and its associated PG is 0.4, let it be written as  $(u_{x_2}, \tilde{u}_{x_2}) = (0.4, \widetilde{0.6})$ . Now, which of the two situations is closer to be a crisp singleton? Which one is considered to have more weight the PG or SG? Which of the two is considered the height? As a matter of fact in some publications, e.g. (Mendel 2001, Mendel & John 2002), the T2FS is depicted with the FOU forming a geometric



base and the SGs acting as the geometric height of the 3D shape representing the T2FS. In the figures adapted in this thesis geometrically the PGs represent the height of the 3D shape. The view adopted here is to define two distinct heights, one for the PG called the *primary height*,  $h(\tilde{A})$ , and the other for the SG called the *secondary height*,  $\tilde{h}(\tilde{A})$ .

**Definition 3.4.3** *Let  $\tilde{A} \in \tilde{F}(X)$  then the primary height,  $h(\tilde{A})$ , is the highest PG attained by any domain value in the set, and the secondary height,  $\tilde{h}(\tilde{A})$ , is the highest SG attained by any domain value at any PG in the set i.e.,*

$$h(\tilde{A}) = \sup_{\forall x} u_x$$

$$\tilde{h}(\tilde{A}) = \sup_{\forall x, \forall u_x} \tilde{u}_x$$

In the case of a reduction from T2FSs to IVFSs or FSs, it is actually the secondary height that is more important. Consulting Table 3.21 the secondary height must be at unity for a T2FS to be reduced to IVFSs or FSs, and both heights must be at unity to reduce to a crisp set. This fact will be very useful in defining normality for T2FSs and its application to type-2 fuzzy numbers. Next, the core and support of T2FSs is defined. The core of FSs is shown in Definition 2.1.9 to be the crisp set that contain the elements of the domain with membership grades at unity. The core represents the elements of the domain that definitely belong to the FS. It can also be represented as shown in Definition 2.1.15 to be the  $\alpha$ -cut at  $\alpha = 1$ .

$$\begin{aligned} core(A) &= \{x \mid A(x) = 1\} \\ &= A_1 \end{aligned}$$

This definition can be extended directly to IVFSs considering its interval membership grade. The following equation is a more restricted definition which considers the interval membership grade to be at unity.

$$\begin{aligned} core(\hat{A}) &= \{x \mid \hat{A}(x) = [1, 1]\} \\ &= \hat{A}_1 \\ &= (\underline{A}_1, \bar{A}_1) \end{aligned} \tag{3.33}$$

In this view, the core of an IVFS can be considered a restrictive definition. Another way of looking at the core is in terms of the height. A more relaxed definition can be associating the core set with the elements that belong to the UMF with height at unity as shown in Figure 3.9. It is the elements of the domain that attain the highest membership grades when the height is at unity. Now consider the core of IVFSs in light of the height definition.

**Definition 3.4.4** Let  $\hat{A} \in \hat{F}(X)$  be an IVFS, then the core  $core(\hat{A})$  of this IVFS is the crisp set that contains the elements of the domain that attain upper membership grades at unity, i.e.,

$$\begin{aligned} core(\hat{A}) &= \{x \mid \bar{A}(x) = 1\} \\ &= \bar{A}_1 \end{aligned}$$

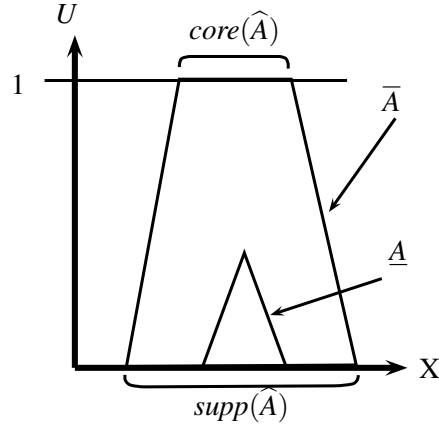


Fig. 3.9. The core,  $core(\hat{A})$ , and support,  $supp(\hat{A})$ , of IVFS  $\hat{A}$ .

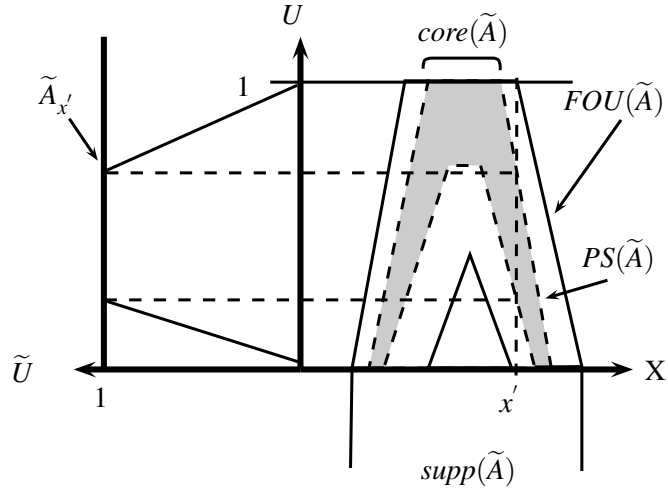


Fig. 3.10. The core,  $core(\tilde{A})$ , and support,  $supp(\tilde{A})$ , of T2FS  $\tilde{A}$ .

In this definition the  $\alpha$ -cut RT is utilised to provide an alternative set-valued definition of the core. It is the  $\alpha$ -cut of the UMF at level  $\alpha = 1$ . In this view, the LMF is not involved at all, it does not have any effect on the definition of the core. This definition is a generalisation of the core provided

in Eq. 3.33. In T2FSs the core will definitely depend on the primary and secondary grades. They should both be at unity to give the semantics of elements that certainly belong to a set in the crisp sense.

**Definition 3.4.5** Let  $\tilde{A} \in \tilde{F}(X)$  then the core,  $core(\tilde{A})$ , is the crisp set that contains the elements of the domain that attain both SGs and PGs at unity, i.e.,

$$core(\tilde{A}) = \{x \mid u_x = 1 \text{ and } \tilde{u}_x = 1\}$$

The core of T2FSs can be achieved by assigning unity to the height of T2FSs, and taking all the elements of the domain that attain a membership grade equal to this height. The  $\alpha$ -cut RT of T2FSs can be used to provide an alternative definition, i.e.,

$$core(\tilde{A}) = \tilde{A}_{1,1} \quad (3.34)$$

In other words, the core of a T2FS is a crisp set that is represented by the  $\alpha$ -cut of T2FS at levels  $\tilde{\alpha} = 1$  and  $\alpha = 1$ . Another description of this equation can be that the core of a T2FS is equal to the core of it's principal set, i.e.,

$$\begin{aligned} core(\tilde{A}) &= core(PS(\tilde{A})) \\ &= core(\tilde{A}_{\tilde{1}}) \\ &= \tilde{A}_{1,1} \end{aligned} \quad (3.35)$$

It is evident since the PS is an IVFS represented by an  $\alpha$ -plane at level  $\tilde{\alpha} = 1$ , then the core of this IVFS is the core of it's UMF. After this investigation into the core, the support of the T2FS is discussed next. In order to start this investigation, the support of FSs is shown following Definition 2.1.8 in which it is defined using membership grades, and Definition 2.1.15 in which it is defined using the  $\alpha$ -cut representation of FSs. Let  $A \in F(X)$  be a FS, then the support of  $A$  is as follows:

$$\begin{aligned} supp(A) &= \{x \mid A(x) > 0\} \\ &= A_{\alpha^+} \end{aligned}$$

Here the meaning conveyed by the support is a crisp set containing all elements that have a degree of membership to the FS. The  $\alpha$ -cut version of the definition uses the strong  $\alpha$ -cut at  $\alpha = 0$ . It can be viewed as the closure of the  $\alpha$ -cut at level  $\alpha = 0$ . Now consider the support of IVFSs in the same manner the core of IVFSs is defined. Here since the support of the LMF is always a subset of the support of the UMF, only the support of the UMF can be used in the definition.

**Definition 3.4.6** Let,  $\hat{A} \in \hat{F}(X)$ . Then, the support,  $supp(\hat{A})$ , is the crisp set that contain the

elements of the domain that attain upper membership grades greater than zero, i.e.,

$$\begin{aligned} \text{supp}(\widehat{A}) &= \{x \mid \overline{A}(x) > 0\} \\ &= \text{supp}(\overline{A}) \\ &= \overline{A}_{\alpha^+} \end{aligned}$$

Strong  $\alpha$ -cuts for IVFSs have not been defined before, and in this definition it is implied that since the UMF is a FS its strong  $\alpha$ -cut is a direct consequence. In T2FSs the support will definitely depend on the primary and secondary grades in the same manner as the core. These grades should both be greater than zero to give the semantics of elements that belong to the T2FS by any degree.

**Definition 3.4.7** *Let,  $\tilde{A} \in \tilde{F}(X)$ . Then, the support,  $\text{supp}(\tilde{A})$ , is the crisp set that contain the elements of the domain that attain both SGs and PGs greater than zero, i.e.,*

$$\text{supp}(\tilde{A}) = \{x \mid u_x > 0 \text{ and } \tilde{u}_x > 0\}$$

In other words, the support of a T2FS is a crisp set that is represented by the strong  $\alpha$ -cut of the FOU of the T2FS. It is evident since the FOU is an IVFS represented by an  $\alpha$ -plane at level  $\tilde{\alpha} = 0$ , then the support of this IVFS is the support of it's UMF.

$$\begin{aligned} \text{supp}(\tilde{A}) &= \text{supp}(\text{FOU}(\tilde{A})) \\ &= \text{supp}(\tilde{A}_{\tilde{0}}) \\ &= \overline{A}_{\tilde{0},0^+} \end{aligned} \tag{3.36}$$

The last line of this equation also has not been defined before, but it follows from the fact that the FOU is an IVFS. The support of the FOU then follows from Definition 3.4.6. Figure 3.10 shows the core and support of a T2FS. Up to this point three important notions have been extended from FSs to IVFSs and T2FSs, the height, the core and the support. Next, the normality of T2FSs is investigated. Normal FSs are used to categorise fuzzy numbers as shown later in this section. A FS  $A \in X$  is normal if  $\exists x_0 \in X$  such that  $A(x_0) = 1$ . It can also be described by requiring the height of the FS equal to unity, i.e.,  $h(A) = 1$ , or by requiring the core of the FS to exist and not be an empty set. This fact can be described using  $\alpha$ -cuts by requiring that the  $\alpha$ -cut at  $\alpha = 1$  is not an empty set, i.e.,  $A_1 \neq \emptyset$  as seen in Definition 2.1.15. In the following, two cases of normal IVFSs are defined separately, one when both the LMF and UMF are normal, and the other when only the UMF is normal.

**Definition 3.4.8 (Perfectly Normal IVFS)** *An IVFS,  $\widehat{A}$ , is said to be perfectly normal if both its LMF and UMF are normal i.e.  $h(\overline{A}) = h(\underline{A}) = 1$ .*

This term has been used in (Kaufmann & Gupta 1985) to describe a perfectly triangular type-2 fuzzy number. This definition can be found in some publications such as Hong & Lee (2002) as part of the definition of an interval valued fuzzy number. This case is definitely more restrictive, and a need for a less restrictive definition is apparent in some applications.

**Definition 3.4.9 (Normal IVFS)** An IVFS,  $\hat{A}$ , is said to be normal if its UMF is normal i.e.  $h(\bar{A}) = 1$ .

In T2FSs a differentiation is made between three cases; normal, semi-perfectly normal, and perfectly normal. Although more than three situations can be named, but it is believed that these three cases are conceptually appealing.

**Definition 3.4.10 (Perfectly Normal T2FS)** A T2FS,  $\tilde{A}$ , is said to be perfectly normal if,  $\exists x_0 \in X$  such that  $\forall u_{x_0} \in J_{x_0}$ ,  $u_{x_0} = 1$  and  $\tilde{A}_{x_0}(u_{x_0}) = 1$ .

This definition means that there is a point in the domain that generalises a crisp number, it is also a generalisation of the perfectly normal IVFS.

**Corollary 3.4.1** Let,  $\tilde{A}$ , be a perfectly normal T2FS, then

- $\tilde{A}_{\tilde{0}}$  and  $\tilde{A}_{\tilde{1}}$  are both perfectly normal.

Indeed, if  $\tilde{A}_{\tilde{1}}$  is a FS (i.e. it is a PMF) then when all the second order uncertainties diminishes a T2FS reduces to a FS represented by its PMF (Karnik & Mendel 2001), hence, the PMF should be normal.

**Definition 3.4.11 (Normal T2FS)** A T2FS,  $\tilde{A}$ , is said to be normal if,  $\exists x_0 \in X$  such that  $\exists u'_{x_0} \in J_{x_0}$  where  $u'_{x_0} = 1$  and  $\tilde{A}_{x_0}(u'_{x_0}) = 1$ .

This definition generalises the perfectly normal case. It still reduces to a normal IVFS when uncertainties disappear. The difference is clear, as it only requires that the second order uncertainties disappear in order to arrive at a crisp number in the perfectly normal case. In the normal case both uncertainties should disappear.

**Corollary 3.4.2** Let,  $\tilde{A}$ , be a normal T2FS, then

- $\tilde{A}_{\tilde{0}}$  and  $\tilde{A}_{\tilde{1}}$  are both normal.

The last case is also a special case of a normal T2FS, but still not perfectly normal. It is the case when the principal set is a normal fuzzy set and the FOU is a normal IVFS. It is useful for some special cases of fuzzy sets such as triangular FSs introduced by Starczewski (2009).

**Definition 3.4.12 (Semi-perfectly Normal T2FS)** A T2FS,  $\tilde{A}$ , is said to be semi-perfectly normal if,  $\exists x_0 \in \tilde{A}$  such that  $\exists u_{x_0}^k, u_{x_0}^l \in J_{x_0}$ ,  $u_{x_0}^k \neq u_{x_0}^l$  where  $u_{x_0}^k = 1 \Rightarrow \tilde{A}_{x_0}(u_{x_0}^k) = 1$  and  $u_{x_0}^l \neq 1 \Rightarrow \tilde{A}_{x_0}(u_{x_0}^l) \neq 1$ .

**Corollary 3.4.3** *Let,  $\tilde{A}$ , be a semi-perfectly normal T2FS, then*

- $\tilde{A}_0$  is normal and  $\tilde{A}_1$  is perfectly normal .

Figure 3.12 depicts these three cases of a normal T2FS.

### 3.4.1 Containment

An important notion that is used in many theoretical investigations is the notion of containment. It is essential in classical set theory to determine whether a set is a subset (contained) of another set or not. As noted in Section 2.1, in this thesis a differentiation is made between containment and subsethood for fuzzy sets and extensions. While they convey the same meaning in classical sets they have different meanings for fuzzy sets. Zadeh (1965) defined the containment for FSs shown in Definition 2.1.12, basically for FSs  $A$  and  $B \in \mathbf{F}(X)$ , then  $A$  is contained in  $B$  can be defined as follows:

$$A \subseteq B \Leftrightarrow A(x) \leq B(x), \forall x \quad (3.37)$$

Accordingly the definition above satisfies the following property:

$$A \subseteq B \Rightarrow A \cup B = B \text{ and } A \cap B = A \quad (3.38)$$

This property is equally satisfied by the subsethood of crisp sets and thus generalises it. Although, the containment between IVFSs appears to be straight forward, it has some aspects to be considered with regards to the semantics of the definition provided in the literature, e.g. (Vlachos & Sergiadis 2007, Zeng & Li 2006a). Let,  $\hat{A}$  and  $\hat{B} \in \hat{\mathbf{F}}(X)$  be two IVFSs then  $\hat{A}$  is contained in  $B$  can be described as follows:

$$\hat{A} \subseteq_1 \hat{B} \Leftrightarrow \underline{A}(x) \leq \underline{B}(x) \text{ and } \overline{A}(x) \leq \overline{B}(x), \forall x \quad (3.39)$$

This definition is shown in Figure 3.13, it is a direct extension to the FS definition. The subscript 1 in  $\subseteq_1$  is used to differentiate between the variety of containment definition presented in the preseding. The inequality is extended from scalar membership grades in FSs to interval membership grades in IVFSs. It can also be seen as the containment between the LMFs and UMFs of these sets, i.e.,

$$\hat{A} \subseteq_1 \hat{B} \Leftrightarrow \underline{A} \subseteq \underline{B} \text{ and } \overline{A} \subseteq \overline{B}$$

this fact allow  $\subseteq_1$  to also extend the property in Eq. 3.38 as follows:

$$\hat{A} \subseteq_1 \hat{B} \Rightarrow \hat{A} \cup \hat{B} = \hat{B} \text{ and } \hat{A} \cap \hat{B} = \hat{A} \quad (3.40)$$

Another expression that could convey the meaning of containment is found in the literature, e.g. (Burillo & Bustince 1996, Bustince et al. 2009). It depends on extending the containment operation from interval arithmetic to the interval membership grades. For example assume that  $A = [\underline{a}, \bar{a}]$  and  $B = [\underline{b}, \bar{b}]$  are two intervals. Then,

$$A \subseteq B \Leftrightarrow \underline{a} \geq \underline{b} \text{ and } \bar{a} \leq \bar{b}$$

This fact is used in the following equation to define containment for IVFSs. Let,  $\hat{A}$  and  $\hat{B} \in \hat{F}(X)$  be two IVFSs. Then,

$$\hat{A} \subseteq_2 \hat{B} \Leftrightarrow \underline{A}(x) \geq \underline{B}(x) \text{ and } \bar{A}(x) \leq \bar{B}(x), \forall x \quad (3.41)$$

This situation is shown in Figure 3.14. Definitely it is not an extension to the FS definition. Using the LMFs and UMFs of these sets, the difference between  $\subseteq_1$  and  $\subseteq_2$  is evident.

$$\hat{A} \subseteq_2 \hat{B} \Leftrightarrow \underline{B} \subseteq \underline{A} \text{ and } \bar{A} \subseteq \bar{B}$$

this fact also implies that  $\subseteq_2$  does not satisfy the property in Eq. 3.40. Semantically, since the whole domain of support of  $\hat{A}$  is contained within  $\hat{B}$ , it does convey the meaning of containment. Moreover, Eq. 3.41 has proven to be useful in some definitions and operations (Burillo & Bustince 1996, Bustince et al. 2009). In Figure 3.15 a situation is introduced that can be conceived as containment between IVFSs and violates both the above mentioned definitions. It is believed that a more relaxed and general definition might be useful for some operations. The following definition is a more flexible containment operation:

**Definition 3.4.13** *Let,  $\hat{A}$  and  $\hat{B} \in \hat{F}(X)$  be two IVFSs. Then,*

$$\hat{A} \subseteq_3 \hat{B} \Leftrightarrow \bar{A}(x) \leq \bar{B}(x), \forall x \quad (3.42)$$

This definition assures that there is no  $x \in \text{supp}(\hat{A})$  that is not contained in  $\text{supp}(\hat{B})$ . This definition also does not satisfy the property in Eq. 3.40. In the case of T2FSs Mizumoto & Tanaka (1976) extended Zadeh's containment in the following manner, let,  $\tilde{A}$  and  $\tilde{B} \in \tilde{F}(X)$  be two T2FSs. Then,

$$\tilde{A} \subseteq_1 \tilde{B} \Leftrightarrow \tilde{A}(x) \leq \tilde{B}(x), \forall x \quad (3.43)$$

since  $\tilde{A}(x) \equiv \tilde{A}_x$  and  $\tilde{B}(x) \equiv \tilde{B}_x$  are FSs, without defining what does the inequality mean this formula will not make any sense. Yang & Lin (2009), defined containment to comply with Zadeh's definition of containment for FSs. Using the same T2FSs  $\tilde{A}$  and  $\tilde{B} \in \tilde{F}(X)$ , then

$$\tilde{A} \subseteq_2 \tilde{B} \Leftrightarrow 0 \leq \tilde{A}_x(u_x) \leq \tilde{B}_x(u_x) \leq 1, \forall x, \forall u_x \quad (3.44)$$

Observe that Yang and Lin's containment does not reduce to Vlachos and Sergiadis's method for IVFSs in case of the special situation when T2FSs reduce to IVFSs. The reason behind that is the lack of involvement of the primary membership grades in the evaluation. Whether this definition satisfies a similar property for those of FSs and IVFSs is an open question. In essence, an alternative definition is proposed based on  $\alpha$ -planes.

**Definition 3.4.14** Let,  $\tilde{A}$  and  $\tilde{B} \in \tilde{F}(X)$  be two T2FSs. Then,

$$\tilde{A} \subseteq_3 \tilde{B} \Leftrightarrow \tilde{A}_{\tilde{\alpha}} \subseteq \tilde{B}_{\tilde{\alpha}}, \forall \tilde{\alpha}$$

One only need to find a suitable IVFS definition to use with each  $\alpha$ -plane. Definition 3.4.13 serves as a flexible definition with each  $\alpha$ -plane. This definition is used with many axioms the measures of uncertainty introduced in Chapter 4. Yet another important definition added to the list of T2FS contributions of this chapter. Next, the important cutworthy property which is directly related to the concept of containment is extended from FS theory to serve both the  $\alpha$ -plane EP and the  $\alpha$ -cut RT.

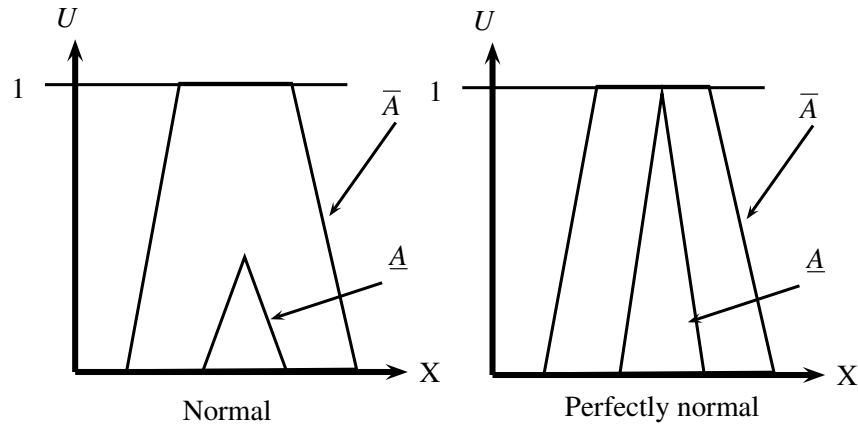


Fig. 3.11. Normal and perfectly normal IVFSs.

### 3.5 Cutworthy property

In Chapter 2, the cutworthy property for FSs is reviewed. It has been shown that a function is cutworthy if it preserves the order of crisp subsethood. It has also been shown that the union and intersection are both cutworthy functions and the negation is not cutworthy. Another issue is recognised regarding the EP in general and the  $\alpha$ -cut version of it. As a matter of fact the EP is



valid for functions and operations that produce output sets of the same type as the input sets, i.e.,

$$f(A_i \in F(X)) = B \in F(Y), i = 1, 2, \dots, n$$

$$f(\widehat{A}_i \in \widehat{F}(X)) = \widehat{B} \in \widehat{F}(Y), i = 1, 2, \dots, n$$

$$f(\widetilde{A}_i \in \widetilde{F}(X)) = \widetilde{B} \in \widetilde{F}(Y), i = 1, 2, \dots, n$$

Note that the result of the union and intersection operations between FSs is another FS. In other words, the EP for FSs, IVFSs, and T2FSs normally serve functions that result in the same type of sets. The same observation is true for the  $\alpha$ -EP for FSs,  $\alpha$ -cut EP for IVFSs, and both the  $\alpha$ -plane EP and  $\alpha$ -cut EP for T2FSs since they are derived using the original EP. The class of functions that can be defined using the  $\alpha$ -plane EP directly should satisfy the cutworthy property defined for the  $\alpha$ -EP for FSs. This leads to the question of how to describe such functions and when can a function be considered a cutworthy function with respect to the  $\alpha$ -plane EP. Another fact is that the EP for IVFSs and T2FSs are derived from the EP of FSs. It follows that the same restrictions imposed by the EP of FSs is extended to those functions acting on IVFSs and T2FSs. This implies that the  $\alpha$ -EP for FSs, IVFSs, and T2FSs share the same class of functions, i.e. cutworthy functions, those that are monotonic and preserve the order of crisp subsethood as explained earlier in Section 2.1.

**Theorem 3.5.1** *Any function that is cutworthy for fuzzy sets, is also cutworthy for interval valued fuzzy sets and type-2 fuzzy sets.*

**Proof.** According to Theorem 3.1.2, the IVFS  $\alpha$ -cut EP is derived from the FS  $\alpha$ -cut EP. The  $\alpha$ -cuts for IVFSs are completely defined by the  $\alpha$ -cuts of two FSs, i.e., the LMF and UMF. This implies that, any function that satisfy the requirements of the FS  $\alpha$ -cut EP also satisfy the FS  $\alpha$ -cut EP requirements. Then if the function is cutworthy for FSs is also cutworthy for IVFSs, and that completes the first part of the proof. The second part is two fold, the first involves the  $\alpha$ -plane EP. To this end, according to Theorem 3.2.1 each  $\alpha$ -plane is derived from the union of the  $\alpha$ -cuts of all the vertical slices, this can be seen in Eq. 2.62. These vertical slices are FSs, and are extended using the  $\alpha$ -cut EP for FSs. It follows that if the function satisfies the requirements of the  $\alpha$ -cut EP for FSs it also satisfy the requirements for the  $\alpha$ -plane EP for T2FSs. Hence, if a function is cutworthy for FSs it is also cutworthy with respect to the  $\alpha$ -planes. The second part involves the  $\alpha$ -cut EP for T2FSs, which is a combination of the  $\alpha$ -plane EP for T2FSs and the IVFS  $\alpha$ -cut EP. Then any function that is cutworthy for the  $\alpha$ -plane EP and the IVFS  $\alpha$ -cut EP is also cutworthy for the  $\alpha$ -cut EP for T2FSs. This means that any function that is cutworthy for FSs is also cutworthy for T2FSs, and that completes the second part of the proof. ■

The question now is how to define functions that need not result in output sets that are the same type as the input sets, e.g., a function that result in a scalar. One of the most important class of functions that result in a scalar are defuzzification functions, e.g. the centroid of FSs. In this

section two main questions are discussed. The first is, how to treat functions that are not cutworthy using the  $\alpha$ -cut RT? The second is, how to treat functions that result in scalars? To answer the first questions, the negation operation is not cutworthy and at the same time the  $\alpha$ -cut representation is used to define negation operations without referring to the  $\alpha$ -EP. In fact,  $(A')_\alpha \neq A'_\alpha$ , which means that the  $\alpha$ -cut of the complement of FS  $A$  is not equal to the complement of the  $\alpha$ -cut of that FS. The  $\alpha$ -cut complement is calculated differently, it is shown in 2.1.3 to be,

$$A'_\alpha = (A_{(1-\alpha)^+})'$$

which is an  $\alpha$ -cut interpretation of the actual point-wise definition of the complementation. To be able to generalise a rule or method for such functions that are not cutworthy is an open question. Meanwhile, a case by case or function by function approach is implemented such as the complementation case. To generalise to IVFSs and T2FSs is also a function specific task. Since, a function is defined for FSs using the  $\alpha$ -cut RT it can easily be extended to IVFSs and T2FSs. As an example the complement operation for IVFSs can be defined as follows:

$$\begin{aligned}\widehat{A}'_\alpha &= (\underline{A}, \overline{A})'_\alpha \\ &= (\overline{A}', \underline{A}')_\alpha \\ &= (\overline{A}'_\alpha, \underline{A}'_\alpha) \\ &= ((\overline{A}_{(1-\alpha)^+})', (\underline{A}_{(1-\alpha)^+})')\end{aligned}\tag{3.45}$$

This definition can also be extended to T2FSs in the same manner using the  $\alpha$ -cut RT.

$$\begin{aligned}\widetilde{A}'_{\tilde{\alpha}, \alpha} &= (\underline{A}, \overline{A})'_{\tilde{\alpha}, \alpha} \\ &= (\overline{A}', \underline{A}')_{\tilde{\alpha}, \alpha} \\ &= (\overline{A}'_{\tilde{\alpha}, \alpha}, \underline{A}'_{\tilde{\alpha}, \alpha}) \\ &= ((\overline{A}_{\tilde{\alpha}, (1-\alpha)^+})', (\underline{A}_{\tilde{\alpha}, (1-\alpha)^+})')\end{aligned}\tag{3.46}$$

There are no closed form formulas for extending functions and operations that are not cutworthy. But as seen in the example of the complementation it is actually easy to find the function for IVFSs and T2FSs as long as a function is present for FSs. To answer the second question, the  $\alpha$ -plane RT is explicitly used in the definition of the centroid of a T2FS by Liu (2008) and Wagner & Hagrass (2008). The centroid of a T2FS,  $\widetilde{A} \in \widetilde{F}(X)$ , is equal to the union of the centroids of its decomposed  $\alpha$ -T2FSs (associated T2FSs), i.e.,

$$c(\widetilde{A}) = \bigcup_{\forall \tilde{\alpha}} c(\widetilde{\alpha} \widetilde{A}_{\tilde{\alpha}})\tag{3.47}$$

Practically the  $\alpha$ -T2FS  $c(\tilde{\alpha}\tilde{A}_{\tilde{\alpha}})$  is treated as IVFS by calculating the centroid of the  $\alpha$ -planes  $c(\tilde{A}_{\tilde{\alpha}})$  using any centroid method for IVFSs (IT2FSs) such as the Karnik-Mendel algorithm or the collapsing method (Greenfield et al. 2009).

$$c(\tilde{A}) = \bigcup_{\forall \tilde{\alpha}} \tilde{\alpha} c(\tilde{A}_{\tilde{\alpha}})$$

In essence this method look very similar to the unary version of  $\alpha$ -plane EP. The difference lie in the nature of the result of calculating  $c(\tilde{A}_{\tilde{\alpha}})$ . The result is an interval, i.e.  $c(\tilde{A}_{\tilde{\alpha}}) \in \mathbf{I}(\mathbf{X})$ , not an interval valued fuzzy set, and given the fact that the centroid of standard FSs is a scalar value gives insight to to be expected as a result from T2FSs. As a matter of fact the centroid of a T2FS is a fuzzy set which is called a type-reduced set. Liu realised this fact and showed that the result from the union operation of all the intervals resulting from the centroid of the  $\alpha$ -planes, is a FS. This FS is represented by the intervals which form an  $\alpha$ -cut representation for the type-reduced set. Liu (2008) suggest that the type-reduced FS can then be processed to reach a real value. This fact can be generalised for a family of functions that result in a fuzzy set rather than a precise number or a T2FS. Hamrawi & Coupland (2009a) defined the non-specificity function using the same concept which can be considered a direct implementation of this theorem. This operation is shown in Hamrawi & Coupland (2010) applied the same concept to some uncertainty measures. It is noticed that a theme is present, for example, when a function is acting on a FS and it produces a single number as its output, its direct extension to IVFSs produce an interval. Furthermore, when the function is extended to T2FSs, the result is a FS. This can be summarised as follows:

$$\begin{aligned} f(A \in \mathbf{F}(\mathbf{X})) &= Y \in \mathfrak{R}^+ \\ f(\hat{A} \in \hat{\mathbf{F}}(\mathbf{X})) &= \mathbf{I}(\mathbf{Y}) \\ f(\tilde{A} \in \tilde{\mathbf{F}}(\mathbf{X})) &= \mathbf{F}(\mathbf{Y}) \end{aligned}$$

In most cases a single number is desired as an outcome, then an operation must be used to reach that number. Popularly the average operation is used for such operations (Wu & Mendel 2007b, Zhai & Mendel 2011). In summary, even if function or operation is not cutworthy, the  $\alpha$ -plane and the  $\alpha$ -cut representation theorems can still be used. In the above investigation some themes can be identified, but no closed formula for functions that are not cutworthy can be defined. Two situation were studied in the aforementioned discussion, these situation will be shown to be very popular in many theoretical and practical applications. The uncertainty measures defined in the following chapter make great use of these class of functions.

### 3.6 Summary

In this chapter, the main contributions of this thesis are presented. This chapter is considered central to all the chapters that follow. The definitions presented in this chapter have the potential of being central to type-2 fuzzy sets as a field, and thus a main contribution to fuzzy sets and systems and related disciplines. The methodology followed in defining the  $\alpha$ -cut representation theorem serve to be followed in other fuzzy set extensions. On the other hand, the  $\alpha$ -cut extension principle for T2FSs have the potential to extend the applicability of the field of type-2 fuzzy logic to many other disciplines, such as generalised information theory as shown in the following chapter. The pattern followed in going from classical sets to intervals to fuzzy sets to interval valued fuzzy sets to type-2 fuzzy sets can be followed to progress to higher types of fuzzy sets such as type-n fuzzy sets. Operations can also be extended in the same progressive manner, and properties can also be defined. This chapter, with the plethora of concepts, definitions and theorems have the potential of being revolutionary to the field of fuzzy sets. What is offered here is not only the results stated in Table 3.22, but a methodology of thinking about fuzzy sets and its different non-standard extensions. All of these sets can break down to being a collection of classical sets related together and restricted by certain conditions. For example, an interval is in fact a continuous classical set, or an uncertain number bound by two values. A fuzzy set on the other hand is a collection of nested crisp sets or intervals related together with level of belonging  $\alpha$ . Interval valued fuzzy sets are a collection of pairs of those nested crisp sets that formulate a bound for the actual set or interval which is not known. Type-2 fuzzy sets represent a collection of these pairs related together with two levels rather than one level of belonging, and vice a versa. Further extension for this idea to other non-standard fuzzy sets may be valid and would be very interesting for further investigation. The new alternative of the  $\alpha$ -cut representation theorem and extension principle for interval valued fuzzy sets are used in defining uncertainty measures in Chapter 4 of this thesis. The following Table 3.22 shows the summary of contribution of this chapter:

Table 3.22. Summary of contributions introduced in this chapter.

Concept	Classification	Reference
<b>Main Results</b>		
$\alpha$ -cut representation theorem for IVFSs	New alternative	Theorem 3.1.1
$\alpha$ -cut extension principle for IVFSs	New alternative	Theorem 3.1.2
$\alpha$ -plane extension principle for T2FSs	Novel	Theorem 3.2.1
$\alpha$ -cut representation theorem for T2FSs	Novel	Theorem 3.3.1
$\alpha$ -cut extension principle for T2FSs	Novel	Theorem 3.3.2
<b>Complementary Results</b>		
Reduction rule	New formalisation	Definition 3.4.1
Height of IVFS	New alternative	Definition 3.4.2
Core of IVFS	New alternative	Definition 3.4.4
Support of IVFS	New alternative	Definition 3.4.6
Normal IVFSs	New alternatives	Definition 3.4.9 and 3.4.8
Containment of IVFSs	New alternative	Definition 3.4.13
Height of T2FS	Novel	Definition 3.4.3
Core of T2FS	Novel	Definition 3.4.5
Support of IVFS	Novel	Definition 3.4.7
Normal T2FSs	Novel	Definition 3.4.11, 3.4.10, and 3.4.12
Containment of T2FSs	Novel	Definition 3.4.14
Cutworthy of T2FSs	Novel discussion	Section 3.5

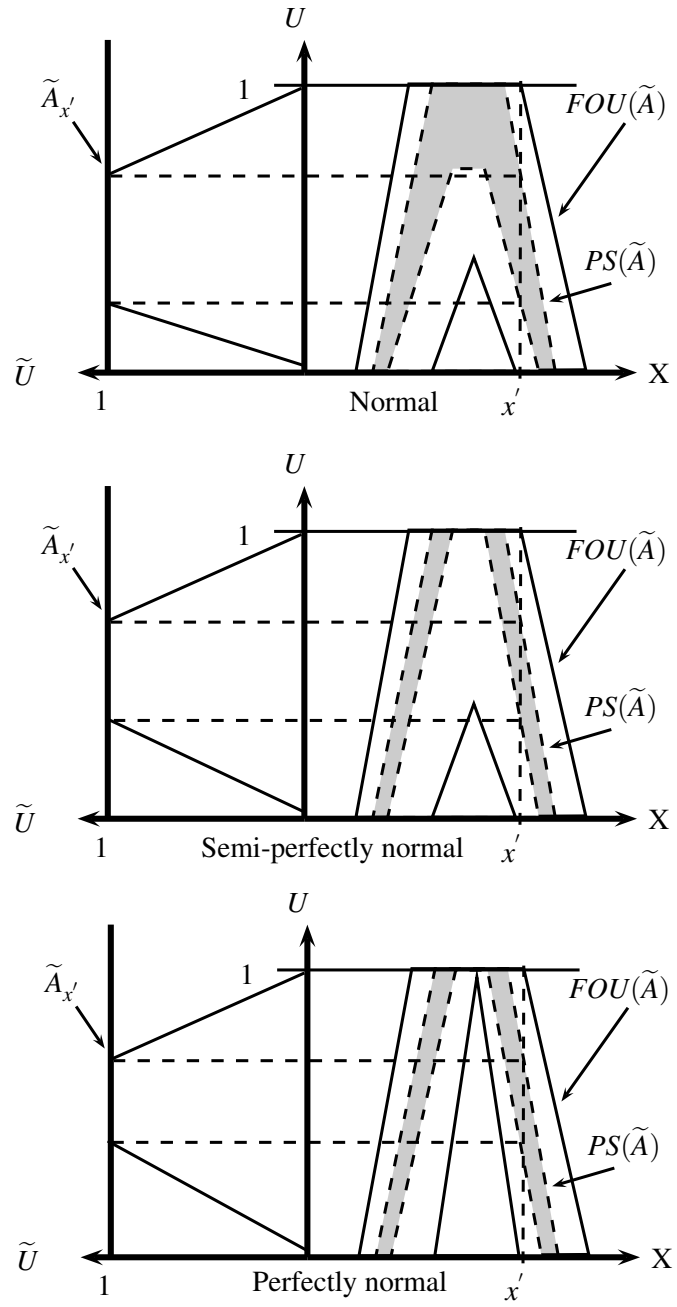


Fig. 3.12. Normal, semi-perfectly normal and perfectly normal T2FSs.

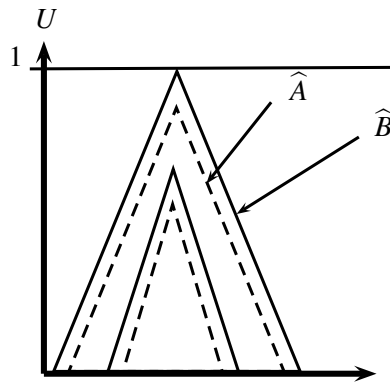


Fig. 3.13.  $\hat{A} \subseteq_1 \hat{B}$ .

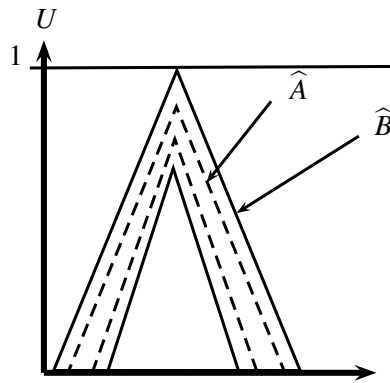


Fig. 3.14.  $\hat{A} \subseteq_2 \hat{B}$ .

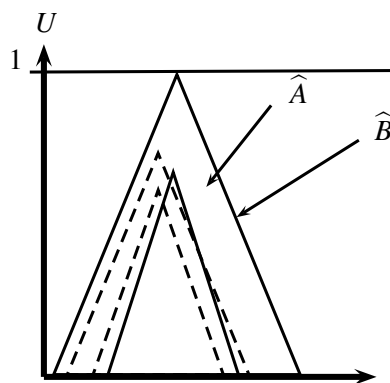


Fig. 3.15.  $\hat{A} \subseteq_3 \hat{B}$ .

## Chapter 4

# Measures of Uncertainty for T2FSs

The Extension Principle based on  $\alpha$ -cuts plays a pivotal role in many applications of fuzzy sets. In the previous chapter two  $\alpha$ -based EPs for T2FSs are defined. One is based on  $\alpha$ -planes in which functions and operations defined for IVFSs can be extended to operate on T2FSs. The other is based on the novel  $\alpha$ -cut representation of T2FSs, in which functions and operations can be extended from crisp sets to T2FSs directly. This chapter explores applications of both the  $\alpha$ -plane EP and the  $\alpha$ -cut EP to define uncertainty measures for T2FSs, in particular cardinality, similarity, subsethood, fuzziness and non-specificity measures. These measures are used as a demonstration of when the  $\alpha$ -plane representation can be used. The choice of uncertainty measures as a means of demonstration is motivated by the centrality of the concept of uncertainty to fuzzy sets in general. Fuzzy logic is advocated to be a field of study greatly involved in handling uncertainties of different aspects (Klir 1991, Klir 2006, Zadeh 2005, Zadeh 2006, Walley 1991, Walley & de Cooman 2001, Shafer 1976). Fuzzy logic's main thesis is to incorporate uncertainty in reasoning and making judgments, as humans do. Uncertainty is an important characteristic in human reasoning and interaction, and critical to the effective modelling of human decision making. Different notions appear from the relation between fuzzy logic and uncertainty measures. Fuzzy logic can be considered as a tool for quantifying uncertainty in different applications, which could be called fuzzy uncertainty measures (Zimmermann 2001). Another approach is to quantify the amount of uncertainty that a fuzzy set exhibits and these can be called uncertainty measures of fuzzy sets, and the latter is the subject of this chapter. Fuzzy sets have a wide range of uncertainty measures in the literature, these measures have been used in many applications (see Zhai & Mendel (2011) for a survey). Uncertainty measures for IVFSs have been widely discussed in the literature (e.g. (Deschrijver & Kral 2007, Wu & Mendel 2007b, Wu 2008, Wu & Mendel 2009)). T2FSs have received less attention, only recently some measures appeared, such as, Cardinality (Jang & Ralescu 2001), Similarity (Hung & Yang 2004, Mitchell 2005, Lin & Yang 2007), and Fuzziness (Al-sharhan et al. 2001). Uncertainty measures for IT2FSs also have been developed by Wu & Mendel (2007b) and Wu & Mendel (2008b) relying heavily on the WSRT. Zhai & Mendel



(2011) defined measures of uncertainty for T2FSs including: centroid, cardinality, fuzziness, variance, and skewness. These measures are defined based on the  $\alpha$ -plane representation of T2FSs without using the  $\alpha$ -plane EP already developed in its first form early in 2009.

## 4.1 Cardinality

Cardinality in crisp set theory represents the number of elements in a given set, and due to the graded membership of elements to given sets in fuzzy set theory it has a different intuitive calculation. It appears many times in the literature of FSs, and there are mainly two types of cardinalities defined: a non-fuzzy cardinality (i.e scalar or interval) and a fuzzy cardinality. The scalar cardinality is a function that gives the result measure in numbers, and the fuzzy cardinality produces a fuzzy set as the measure. Here the main interest is in scalar cardinality, because it is less complicated and gives results that can be compared directly to other measures of uncertainty such as probabilistic measures. Scalar cardinalities have been addressed several times in the literature of FSs (see Wygralak (2000) for a survey). An axiomatic approach has been formulated by Wygralak (2000). Given fuzzy sets  $A, B \in \mathbf{F}(X)$ , elements  $x, y \in X$  and  $a, b \in [0, 1]$ , then the scalar cardinality  $sc$  is a function  $sc : \mathbf{F}(X) \rightarrow \mathfrak{R}^+$  that satisfies the following axioms:

- **(SC 1):**  $a \leq b \Rightarrow sc(x, a) \leq sc(y, b)$  (Singleton monotonicity).
- **(SC 2):**  $supp(A) \cap supp(B) = \emptyset \Rightarrow sc(A \cup B) = sc(A) + sc(B)$ .
- **(SC 3):**  $A$  is crisp  $\Rightarrow sc(A) = |supp(A)|$ , where  $|supp(A)|$  is the classical cardinality defined on crisp sets, i.e., if  $A \in \mathbf{C}(X)$  is a crisp set then  $|A|$  is equal to the number of elements in  $A$  if it is discrete, or the width of  $A$  if it is an interval, i.e., if  $A = [\underline{a}, \bar{a}]$  then  $\|A\| = \bar{a} - \underline{a}$ .

The postulates above proved to introduce several properties and generalise most of the definitions of scalar cardinality presented in the literature. Deschrijver & Kral (2007) extend these axioms to define the cardinality of IVFSs. Let  $\hat{A} \in \hat{\mathbf{F}}(X)$ , and  $sc : \hat{\mathbf{F}}(X) \rightarrow \mathfrak{R}^+$ , then the following are the axioms of cardinality for IVFSs:

- **(SC 1):**  $\hat{A} \subseteq \hat{B} \Rightarrow sc(\hat{A}) \leq sc(\hat{B})$
- **(SC 2):**  $supp(\hat{A}) \cap supp(\hat{B}) = \emptyset \Rightarrow sc(\hat{A} \cup \hat{B}) = sc(\hat{A}) + sc(\hat{B})$
- **(SC 3):**  $\hat{A}$  is crisp  $\Rightarrow sc(\hat{A}) = |supp(\hat{A})|$

The support of an IVFSs is stated earlier in Section 3.4 to be the support of the UMF, i.e.,  $|supp(\hat{A})| = |supp(\bar{A})|$ . The following definition for cardinality of T2FSs is presented in order to extend the axiomatic systems of FSs and IVFSs to T2FSs.

**Definition 4.1.1** A real function  $sc : \tilde{F}(X) \rightarrow \mathfrak{R}^+$  is a cardinality measure for T2FSs, if  $\tilde{A}, \tilde{B} \in \tilde{F}(X)$ , and  $sc$  satisfies the following properties:

- (SC 1):  $\tilde{A} \subseteq \tilde{B} \Rightarrow sc(\tilde{A}) \leq sc(\tilde{B})$
- (SC 2):  $supp(\tilde{A}) \cap supp(\tilde{B}) = \emptyset \Rightarrow sc(\tilde{A} \cup \tilde{B}) = sc(\tilde{A}) + sc(\tilde{B})$
- (SC 3): If  $\tilde{A}$  is crisp  $\Rightarrow sc(\tilde{A}) = |supp(\tilde{A})|$

A natural way of defining this measure for T2FSs is to use the T2FS  $\alpha$ -EP based on  $\alpha$ -planes, i.e.,

$$sc(\tilde{A}) = \bigcup_{\forall \alpha} \alpha sc(\tilde{A}_{\alpha})$$

To be able to use this kind of extension one must make sure that this function is cutworthy, and that is obvious from axiom (SC 1). A meaningful result would be to choose  $sc(\tilde{A}_{\alpha})$  to result in a scalar number and then take the average of all the  $\alpha$ -planes.

**Theorem 4.1.1** Given a T2FS,  $\tilde{A} \in \tilde{F}(X)$ , then the average of the cardinality measures of its decomposed  $\alpha$ -planes is a cardinality measure for T2FSs called average cardinality  $sc(\tilde{A})$ , i.e.,

$$sc(\tilde{A}) = \frac{\sum_{j=1}^M sc(\tilde{A}_{\alpha_j})}{M}$$

where  $M$  is the number of  $\alpha$ -planes.

**Proof.** Check that this measure satisfies the axioms above.

SC 1: Since,

$$\tilde{A} \subseteq \tilde{B} \Rightarrow \tilde{A}_{\alpha} \subseteq \tilde{B}_{\alpha}, \forall \alpha$$

and

$$sc(\tilde{A}_{\alpha}) \leq sc(\tilde{B}_{\alpha}), \forall \alpha$$

from SC 1 of the axioms of the cardinality of IVFSs, hence  $sc(\tilde{A}) \leq sc(\tilde{B})$ .

SC 2: Since

$$\tilde{A} \cap \tilde{B} = \emptyset \Rightarrow \tilde{A}_{\alpha} \cap \tilde{B}_{\alpha} = \emptyset, \forall \alpha$$

and

$$sc(\tilde{A}_{\alpha} \cup \tilde{B}_{\alpha}) = sc(\tilde{A}_{\alpha}) + sc(\tilde{B}_{\alpha}), \forall \alpha$$

then

$$sc(\tilde{A} \cup \tilde{B}) = sc(\tilde{A}) + sc(\tilde{B})$$

SC 3: it is straight forward from the definition. ■

These cardinality measures are very useful for applications.

## 4.2 Similarity Measures

The extent of a given fuzzy sets similarity to another fuzzy set is determined by a function called the similarity measure. Liu (1992) defined the axioms of the distance and similarity measures for FSs, which are widely accepted. Zeng & Li (2006a) and Wu & Mendel (2009) defined the similarity axioms for IVFSs as follows:

- (S 1):  $s(\widehat{A}, \widehat{B}) = s(\widehat{B}, \widehat{A})$
- (S 2):  $s(\widehat{A}, \widehat{A}') = 0$  iff  $\widehat{A}$  is crisp
- (S 3):  $s(\widehat{A}, \widehat{B})$  is maximum iff  $\widehat{A} = \widehat{B}$
- (S 4): if  $\widehat{A} \subseteq \widehat{B} \subseteq \widehat{C}$ , then  $s(\widehat{A}, \widehat{B}) \geq s(\widehat{A}, \widehat{C})$  and  $s(\widehat{B}, \widehat{C}) \geq s(\widehat{A}, \widehat{C})$

Hung & Yang (2004) provided the first definition of T2FS similarity measures generalising the axioms of Liu (1992).

**Definition 4.2.1** A real function  $s : \widetilde{F}(X) \times \widetilde{F}(X) \rightarrow \mathfrak{R}^+$  is a similarity measure between T2FSs, if  $s$  satisfies the following properties

- (S 1):  $s(\widetilde{A}, \widetilde{B}) = s(\widetilde{B}, \widetilde{A})$
- (S 2):  $s(\widetilde{A}, \widetilde{A}') = 0$  iff  $\widetilde{A}$  is crisp
- (S 3):  $s(\widetilde{A}, \widetilde{B})$  is maximum iff  $\widetilde{A} = \widetilde{B}$
- (S 4): if  $\widetilde{A} \subseteq \widetilde{B} \subseteq \widetilde{C}$ , then  $s(\widetilde{A}, \widetilde{B}) \geq s(\widetilde{A}, \widetilde{C})$  and  $s(\widetilde{B}, \widetilde{C}) \geq s(\widetilde{A}, \widetilde{C})$

A normalised measure will ensure that  $s \in [0, 1]$ . Now the T2FS  $\alpha$ -EP based on  $\alpha$ -planes is used to extend this definition to T2FSs.

$$s(\widetilde{A}, \widetilde{B}) = \bigcup_{\forall \alpha} \alpha s(\widetilde{A}_{\alpha}, \widetilde{B}_{\alpha})$$

It is obvious that  $s(\widetilde{A}_{\alpha}, \widetilde{B}_{\alpha})$  is cutworthy from (S 4) of the axioms of similarity of IVFSs. It is useful to define an average similarity as follows:

**Theorem 4.2.1** Given a T2FSs,  $\widetilde{A}, \widetilde{B} \in \widetilde{F}(X)$ , then the average of the similarity measures between their decomposed  $\alpha$ -planes is a similarity between T2FSs called average similarity  $s(\widetilde{A}, \widetilde{B})$ , i.e.,

$$s(\widetilde{A}, \widetilde{B}) = \frac{\sum_{j=1}^M s(\widetilde{A}_{\alpha_j}, \widetilde{B}_{\alpha_j})}{M}$$

where  $M$  is the number of  $\alpha$ -planes.

**Proof.** Check that this measure satisfies the similarity axioms above.

S 1: it is trivial since

$$s(\tilde{A}_{\tilde{\alpha}}, \tilde{B}_{\tilde{\alpha}}) = s(\tilde{B}_{\tilde{\alpha}}, \tilde{A}_{\tilde{\alpha}}), \forall \tilde{\alpha}$$

S 2: also trivial since

$$s(\tilde{A}_{\tilde{\alpha}}, (\tilde{A}_{\tilde{\alpha}})') = 0, \forall \tilde{\alpha}$$

if they were crisp, and hence  $\frac{0}{M} = 0$

S 3: it is also trivial since

$$s(\tilde{C}_{\tilde{\alpha}}, \tilde{C}_{\tilde{\alpha}}) = \max_{\tilde{A}_{\tilde{\alpha}}, \tilde{B}_{\tilde{\alpha}}} s(\tilde{A}_{\tilde{\alpha}}, \tilde{B}_{\tilde{\alpha}}), \forall \tilde{C}_{\tilde{\alpha}}, \forall \tilde{\alpha}$$

S 4: it is also trivial since

$$\tilde{A}_{\tilde{\alpha}} \subseteq \tilde{B}_{\tilde{\alpha}} \subseteq \tilde{C}_{\tilde{\alpha}}$$

, then

$$s(\tilde{A}_{\tilde{\alpha}}, \tilde{B}_{\tilde{\alpha}}) \geq s(\tilde{A}_{\tilde{\alpha}}, \tilde{C}_{\tilde{\alpha}})$$

and

$$s(\tilde{B}_{\tilde{\alpha}}, \tilde{C}_{\tilde{\alpha}}) \geq s(\tilde{A}_{\tilde{\alpha}}, \tilde{C}_{\tilde{\alpha}}), \forall \tilde{\alpha}$$

and that completes the proof. ■

Similarity measures are used extensively in the literature.

#### 4.2.1 Subsethood

The subsethood (or inclusion) measure is a function that determines to what extent is a fuzzy set included in another fuzzy set. Bustince et al. (2006) studied and provided a survey of subsethood functions including the axioms below. Let  $i : F(X) \times F(X) \rightarrow \mathfrak{R}^+$  and  $A, B \in F(X)$  Then  $i$  is a measure of subsethood if it satisfies the following axioms:

- **(I 1):**  $A \subseteq B \Rightarrow i(A, B) = 1$
- **(I 2):**  $i(A, A) = 1$
- **(I 3):** if  $A \subseteq B \subseteq C$  then  $i(C, A) \leq i(B, A)$ , and  $i(C, A) \leq i(C, B)$

Also Vlachos & Sergiadis (2007) defined measures for IVFSs and extended the work by Kosko (1990) to IVFSs.

- **(I 1):**  $\hat{A} \subseteq \hat{B} \Rightarrow i(\hat{A}, \hat{B}) = 1$
- **(I 2):**  $i(\hat{A}, \hat{A}) = 1$

- **(I 3):** if  $\widehat{A} \subseteq \widehat{B} \subseteq \widehat{C}$  then  $i(\widehat{C}, \widehat{A}) \leq i(\widehat{B}, \widehat{A})$ , and  $i(\widehat{C}, \widehat{A}) \leq i(\widehat{C}, \widehat{B})$

Rickard et al. (2009) defined subsethood measures for type-n fuzzy sets which includes IVFSs and T2FSs based on the WSRT. Yang & Lin (2009) extended the axioms of subsethood to T2FSs as follows:

**Definition 4.2.2** A real function  $i : \widetilde{F}(X) \times \widetilde{F}(X) \rightarrow \mathfrak{R}^+$  is a subsethood measure between T2FSs, if  $i$  satisfies the following axioms

- **(I 1):**  $\widetilde{A} \subseteq \widetilde{B} \Rightarrow i(\widetilde{A}, \widetilde{B}) = 1$
- **(I 2):**  $i(\widetilde{A}, \widetilde{A}) = 1$
- **(I 3):** if  $\widetilde{A} \subseteq \widetilde{B} \subseteq \widetilde{C}$  then  $i(\widetilde{C}, \widetilde{A}) \leq i(\widetilde{B}, \widetilde{A})$ , and  $i(\widetilde{C}, \widetilde{A}) \leq i(\widetilde{C}, \widetilde{B})$

Here the T2FS  $\alpha$ -EP based on  $\alpha$ -planes is used to extend this definition to T2FSs.

$$i(\widetilde{A}, \widetilde{B}) = \bigcup_{\forall \widetilde{\alpha}} \widetilde{\alpha} i(\widetilde{A}_{\widetilde{\alpha}}, \widetilde{B}_{\widetilde{\alpha}}) \quad (4.1)$$

It is only needed to show that  $i(\widetilde{A}, \widetilde{B})$  is cutworthy, and that is obvious from axiom **(I 3)**. It is useful to define an average subsethood as follows:

**Theorem 4.2.2** Given a T2FSs,  $\widetilde{A}, \widetilde{B} \in \widetilde{F}(X)$ , then the average of the subsethood measures between their decomposed  $\alpha$ -planes is a subsethood between T2FSs called average subsethood  $i(\widetilde{A}, \widetilde{B})$ , i.e.,

$$i(\widetilde{A}, \widetilde{B}) = \frac{\sum_{j=1}^M i(\widetilde{A}_{\widetilde{\alpha}_j}, \widetilde{B}_{\widetilde{\alpha}_j})}{M}$$

where,  $M$ , is the number of  $\alpha$ -planes.

**Proof.** Check that this measure satisfies the subsethood axioms above, which is as straight forward as the similarity measure. ■

It is also useful to point out the relation between subsethood and similarity (Yang & Lin 2009), i.e., the subsethood is represented as follows

$$i(\widetilde{A}, \widetilde{B}) = s(\widetilde{A}, \widetilde{A} \cap \widetilde{B}) \quad (4.2)$$

### 4.3 Fuzziness

One might be interested in the question of how fuzzy (i.e. uncertain) is the fuzzy set?. The first quantification of the amount of uncertainty associated with a FS was proposed by Zadeh (1968) using the Entropy of a probability distribution. Several proposals were presented to quantify the

fuzziness in FSs and IVFSs, extensive surveys can be found in (Klir & Folger 1988, Al-sharhan et al. 2001, Wu & Mendel 2007b). Some axioms have been stated by De Luca & Termini (1972) for function  $f_z$  to qualify as a fuzziness measure.

- **(FZ 1):**  $A$  is crisp  $\Rightarrow f_z(A) = 0$
- **(FZ 2):**  $f_z(A)$  is maximum  $\Rightarrow A(x) = 0.5, \forall x$
- **(FZ 3):**  $A \leq B \Rightarrow f_z(A) \leq f_z(B)$
- **(FZ 4):**  $f_z(A) = f_z(A')$

where  $A \leq B$  means  $A$  is sharper than  $B$  which is defined for FSs as follows

$$A \leq B = \begin{cases} A(x) \leq B(x), & B(x) \leq 0.5 \\ A(x) \geq B(x), & B(x) \geq 0.5 \end{cases} \quad (4.3)$$

(FZ 1) says that when  $A$  is crisp then the fuzziness is zero (i.e. no fuzziness), (FZ 2) shows that  $f_z$  is more towards the middle which is a unique maximum (FZ 3) ensures that the further away from 0.5 the less fuzzy the set, and (FZ 4) the fuzziness of the set and its complement are equal. The generalised class of measures of fuzziness for FSs are defined by Klir & Folger (1988) to be,

$$f(A) = h\left(\sum_{x \in X} g_x(A(x))\right) \quad (4.4)$$

where  $g_x : [0, 1] \rightarrow \mathfrak{R}^+$ , which is monotonically increasing in  $[0, 0.5]$  and monotonically decreasing in  $[0.5, 1]$ ,  $g_x(0.5)$  is a unique maximum, and  $h$  is monotonically increasing. This definition is also valid for a host of distance measures. For IVFSs these axioms has been modified slightly as follows: (Vlachos & Sergiadis 2007, Zeng & Li 2006a)

- **(FZ 1):**  $\hat{A}$  is crisp  $\Rightarrow f_z(\hat{A}) = 0$
- **(FZ 2):**  $f_z(\hat{A})$  is maximum  $\Rightarrow \underline{A}(x) + \bar{A}(x) = 1 \forall x$
- **(FZ 3):**  $\hat{A} \leq \hat{B} \Rightarrow f_z(\hat{A}) \leq f_z(\hat{B})$
- **(FZ 4):**  $f_z(\hat{A}) = f_z(\hat{A}')$

where  $\hat{A} \leq \hat{B}$  means  $\hat{A}$  is sharper than  $\hat{B}$ , i.e.,

$$\hat{A} \leq \hat{B} = \begin{cases} \underline{A}(x) \leq \underline{B}(x) \text{ and } \bar{A}(x) \leq \bar{B}(x), & \underline{B}(x) + \bar{B}(x) \leq 1 \\ \underline{A}(x) \geq \underline{B}(x) \text{ and } \bar{A}(x) \geq \bar{B}(x), & \underline{B}(x) + \bar{B}(x) \geq 1 \end{cases}$$

Wu & Mendel (2007b) defined a fuzziness measure for IVFSs making use of the WSRT, the fuzziness  $fz(\hat{A})$  is defined as follows:

$$fz(\hat{A}) = [fz(\underline{A}^*), fz(\overline{A}^*)] \quad (4.5)$$

where

$$\underline{A}^*(x) = \begin{cases} \overline{A}(x), & \overline{A}(x) \text{ is further away 0.5 than } \underline{A}(x) \\ \underline{A}(x), & \text{Otherwise} \end{cases}$$

and

$$\overline{A}^*(x) = \begin{cases} \overline{A}(x), & \text{if both } \underline{A}(x) \text{ and } \overline{A}(x) \text{ are below 0.5} \\ \underline{A}(x), & \text{if both } \underline{A}(x) \text{ and } \overline{A}(x) \text{ are above 0.5} \\ 0.5, & \text{Otherwise} \end{cases}$$

and  $fz(A)$  defined by equation (4.4). Intuitively Yager (1979) defined the distinction between a fuzzy set and its complement to be a fuzziness measure, this conception relates distance measures and cardinality to fuzziness. The axioms of fuzziness for T2FSs can be defined as follows

**Definition 4.3.1** A real function  $fz : \tilde{F}(X) \rightarrow \mathfrak{R}^+$  is a fuzziness measure for T2FSs, if  $fz$  satisfies the following properties

- (FZ 1):  $\tilde{A} \text{ is crisp} \Rightarrow fz(\tilde{A}) = 0$
- (FZ 2):  $fz(\tilde{A}) \text{ is maximum} \Rightarrow \tilde{A}(x) \text{ is maximum fuzzy, } \forall x.$
- (FZ 3):  $\tilde{A} \leq \tilde{B} \Rightarrow fz(\tilde{A}) \leq fz(\tilde{B})$
- (FZ 4):  $fz(\tilde{A}) = fz(\tilde{A}')$

In this definition the second and third axioms need further investigation. Axiom (FZ 2) is suggesting a situation where  $\tilde{A}$  attains maximum fuzziness and thus the measure  $fz$  reaches its maximum (e.g. 1 if it is normalised). In T2FS context a set will attain its maximum fuzziness when all the memberships are maximum fuzzy, in other words, all the vertical slices are maximum fuzzy, this can be represented mathematically as follows. Let  $\tilde{A} \in \tilde{F}(X)$ , and maximum fuzziness be  $fz_{\max}$ , then

$$fz_{\max}(\tilde{A}(x)) \equiv fz_{\max}(\tilde{A}_x) \quad (4.6)$$

since  $\tilde{A}_x \in F([0,1])$ , then

$$fz_{\max}(\tilde{A}) \Rightarrow \tilde{A}_x(u_x) = 0.5, \forall u_x, \forall x \quad (4.7)$$

Now it is straight forward to extend the definition of sharpness

**Definition 4.3.2** A T2FS  $\tilde{A}$  is said to be sharper than  $\tilde{B}$  if the following is true

$$\tilde{A} \leq \tilde{B} = \begin{cases} \tilde{A}_x(u_x) \leq \tilde{B}_x(u_x), & \tilde{B}_x(u_x) \leq 0.5 \\ \tilde{A}_x(u_x) \geq \tilde{B}_x(u_x), & \tilde{B}_x(u_x) \geq 0.5 \end{cases}, \forall u_x, \forall x$$

It is clear that fuzziness can not be defined directly because by definition this measure is not cutworthy. This is clear from the axioms, as there is not an axiom that indicates the fuzziness is cutworthy. It is also useful to point out the relation between fuzziness and similarity discussed by Zeng & Li (2006a), i.e., the fuzziness is represented as the similarity between a set and its complement. Which is true for all fuzzy sets by definition. This definition can be extended to T2FSs directly as follows:

$$fz(\tilde{A}) = s(\tilde{A}, \tilde{A}') \quad (4.8)$$

The similarity can be calculated as demonstrated in the previous section, but the complement is surely not cutworthy (Klir & Folger 1988). The easiest way is to calculate the complement using another method then take the similarity of the two sets.

## 4.4 Non-specificity Measures

Klir & Folger (1988), identified that uncertainty is categorised into two main streams, *Ambiguity* and *Vagueness*. Ambiguity results from the lack of distinction among alternatives of the accepted or genuine alternative which leads to two types, *Non-specificity*, describing lack of certain distinctions characterising an object, and *Strife*, describing conflicting distinctions. Vagueness results from the lack of sharp boundaries of relevant alternatives. Within the frame work of fuzzy set theory measures of two of these kinds of uncertainty are defined, *non-specificity*, which relates to sizes (cardinalities) of relevant sets of alternatives, and *fuzziness* (vagueness), which relates to the imprecise boundaries of fuzzy sets. Non-specificity and its counterpart, specificity, are thoroughly discussed in the FS literature (Klir & Folger 1988, Klir & Yuan 1995, Klir 2006, Yager 1982, Yager 2008a), and they are used in many applications including computing with words (CWW) (Klir 2006, Yager 2004).

### 4.4.1 Non-specificity for Fuzzy Sets

Non-specificity for a FS,  $A \in F(X)$ , has been discussed in (Klir & Folger 1988, Klir & Yuan 1995, Klir 2006) as a natural generalisation of the Hartley function for crisp sets.

$$H(A) = \log_2 \|A\| \quad (4.9)$$



where  $A \in \mathbf{C}(X)$  is a crisp set,  $\|A\|$  is the cardinality or simply the length if  $A$  is an interval, and  $\log_2$  gives the result in *bits*. The Generalised Hartley (hereafter called non-specificity) measure for convex FSs can be found in Klir & Yuan (1995), and defined in equation (4.10).

$$ns(A) = \frac{1}{h_A} \int_0^{h_A} \log_2 [1 + \|A_\alpha\|] .d\alpha \quad (4.10)$$

where  $A_\alpha$  is the  $\alpha$ -cut of FS  $A \in \mathbf{F}(X)$ ,  $\|A_\alpha\|$  is the cardinality of the  $\alpha$ -cut,  $h_A$  is the height of the FS, and  $\int$  is a Lebesgue integral. This measure is shown to be unique and satisfies the following requirements:

- **(NS1):**  $ns(A) \in [0, \infty+)$
- **(NS2):**  $A = \{x\} \Rightarrow ns(A) = 0$
- **(NS3):**  $A \subseteq B \Rightarrow ns(A) \leq ns(B)$

The first and second requirements are boundary conditions that shows that the measure of non-specificity produces positive reals and that the only set that with a non-specificity of zero is the singleton. The third requirement is monotonicity. Other properties of this non-specificity measures proved to be unique which is out of the scope of this study (for more details refer to Klir (2006)). A normalised version, given in equation (4.11) taken from Martin & Klir (2006), will ensure  $ns(A) \in [0, 1]$  and axiom (NS1) will be:

- **(NS1):**  $ns(A) \in [0, 1]$

The following equation satisfies this property:

$$ns_K(A) = \frac{1}{h_A} \frac{\int_0^{h_A} \log_2 [1 + \|A_\alpha\|] .d\alpha}{\log_2(1 + \|X\|)} \quad (4.11)$$

This normalised form of the non-specificity measure has been used in CWW context by Martin & Klir (2006) to measure the amount of information in a FS.

$$i(A) = 1 - ns_K(A) \quad (4.12)$$

Dubois & Prade (1999) noticed that  $1 - ns_K(A)$  satisfy the specificity (certainty) axioms, put forward by (Yager 1982).

- **(SP1):**  $\forall A, sp(A) \in [0, 1]$
- **(SP2):**  $sp(A) = 1$  iff  $A = \{x\}$
- **(SP3):**  $A \subseteq B \Rightarrow sp(A) \geq sp(B)$

(Yager 1982) and Yager (2008a) discussed different functions that satisfies these requirements for FSs, e.g.

$$sp_Y(A) = h_A - \frac{1}{\|X\|} \int_0^{h_A} \|A_\alpha\| d\alpha \quad (4.13)$$

and Garmendia et al. (2003) generalised these measures using a T-specificity measure as given in equation (4.14).

$$sp_G(A) = T_1(\mu_A(x_1), N(S_{j=2,\dots,n}(T_2(\mu_A(x_j), w_j)))) \quad (4.14)$$

where  $T_1, T_2$  are t-norms,  $S_{j=2,\dots,n}$  is an  $n$  argument t-conorm,  $N$ , is the negation, and  $w_j$ , is a weighting vector. It is also noted that  $1 - sp(A)$  satisfies axioms (NS1-NS3) and hence are measures of non-specificity. The non-specificity measure,  $ns_k$ , is a measure of how much a set,  $A$ , lacks to be fully informative, in other words, how much is needed for this set to be specific (certain) with reference to its singleton. In figure (4.1), FSs  $A$  and  $B$  are both normal. Intuitively, it is clear that the most specific state would be at the singleton,  $C$ , and its non-specificity should be zero. It is also clear that  $A$  is more non-specific than  $B$ . The specificity measure,  $sp_Y$ , also compares the set to the singleton by measuring how close the set is from being a singleton.

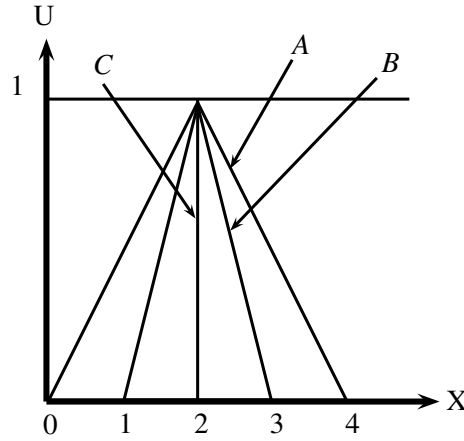


Fig. 4.1.  $A$  and  $B$  are FSs and  $C$  is a crisp singleton

#### 4.4.2 Non-specificity for IVFSs

To examine the extension of the measures of non-specificity to IVFSs, the axiomatic requirements of non-specificity are themselves extended to IVFSs. A non-specificity measure for an IVFS should satisfy the following

- (NS1):  $ns(\hat{A}) \in [0, \infty+)$
- (NS2):  $\hat{A} = \{x\} \Rightarrow ns(\hat{A}) = 0$

- **(NS3):**  $\hat{A} \subseteq \hat{B} \Rightarrow ns(\hat{A}) \leq ns(\hat{B})$ .

These basic requirements provide the same intuitions as the FS non-specificity axioms. There is a particular importance of axiom **(NS2)** that need to be emphasised. This restriction states that the only case which is totally specific is the singleton. Some may consider in the case of IVFSs that a specific case is when these sets reduce to FSs. Frankly speaking this assertion is semantically incorrect. Indeed, the FS is still non-specific to some degree and the crisp singleton can be the point of reference. Turksen (1996) defined a non-specificity measure for IVFS at a domain value,  $x'$ , as follows:

$$ns_T(x') = \left( -\log_{10}(\|\hat{A}(x')\|) \right)^{-1} \quad (4.15)$$

where  $\hat{A}(x') = [\underline{A}(x'), \bar{A}(x')]$  and hence,  $\|\hat{A}(x')\| = \bar{A}(x') - \underline{A}(x')$ . This is not a measure for non-specificity on set,  $\hat{A}$ , but only on one domain value,  $x'$ . This concept can be used to define non-specificity for the complete domain. At each domain value,  $x_i, i = 1, 2, \dots, n$ ,

$$ns_T(x_i) = \left( -\log_{10}(\|\hat{A}(x_i)\|) \right)^{-1} \quad (4.16)$$

then the accumulation of the non-specificity for,  $\hat{A}$ , can be seen as the summation of the non-specificity for all domain values

$$ns_T(\hat{A}) = \sum_{i=1}^n \left( -\log_{10}(\|\hat{A}(x_i)\|) \right)^{-1} \quad (4.17)$$

and for a continuous domains

$$ns_T(\hat{A}) = \int_{x_1}^{x_n} \left( -\log_{10}(\|\hat{A}(x)\|) \right)^{-1} .dx \quad (4.18)$$

Turksen's function will not provide the results in a standard well known format such as the convenient *bits* used by Klir. Using the Generalised Hartley function Turksen's measure of non-specificity can be redefined as follows:

$$ns_H(\hat{A}) = \frac{1}{x_n - x_1} \int_{x_1}^{x_n} \log_2(\|1 + \hat{A}(x)\|) .dx \quad (4.19)$$

The non-specificity measures,  $ns_T(\hat{A})$  and  $ns_H(\hat{A})$ , measure how much is a domain value's interval membership function is non-specific, hence, this function does not distinguish between a FS case and a singleton one, which clearly violates the second requirement. Consider Figure 4.2 which shows an interval membership grade with a non-specificity which does not relate to the non-specificity of the overall set. Instead, an extension of the Hartley based non-specificity measure,  $ns(A)$  or  $ns_K(A)$ , to IVFSs which gives a non-specificity measure which satisfies axioms NS1,

NS2, and NS3. Early versions of these definitions are defined in a conference paper by Hamrawi & Coupland (2009a).

**Theorem 4.4.1** *Given a IVFS,  $\hat{A} \in \hat{F}(X)$ , then the following is a non-specificity measure for IVFSs, i.e.,*

$$ns(\hat{A}) \equiv \bigcup_{\forall A_e} ns(A_e) = [ns(A_e), \overline{ns}(A_e)] \quad (4.20)$$

where,

$$\underline{ns}(A_e) = \min_{\forall A_e} ns(A_e) = ns(\underline{A}) \quad (4.21)$$

and

$$\overline{ns}(A_e) = \max_{\forall A_e} ns(A_e) = ns(\overline{A}) \quad (4.22)$$

**Proof.** Check if the function (measure) satisfies the axioms as follows:

Axioms NS1 and NS2:

By definition,  $ns(\hat{A}) = [ns(\underline{A}), ns(\overline{A})]$ , since  $\underline{A}$  and  $\overline{A}$  are FSs, then from (NS1)  $ns(\underline{A}) \in [0, \infty+)$  and  $ns(\overline{A}) \in [0, \infty+)$ , and hence,  $[ns(\underline{A}), ns(\overline{A})] \in [0, \infty+)$ . From (NS2),

$$\underline{A} = \overline{A} = \{x\} \Rightarrow ns(\underline{A}) = ns(\overline{A}) = 0$$

hence,

$$\hat{A} = \{x\} \Rightarrow [ns(\underline{A}), ns(\overline{A})] = 0$$

The proof of axiom NS3 follows directly from axiom NS3 for FSs. ■

There is a very interesting observation, which is the fact that the non-specificity of IVFSs is an

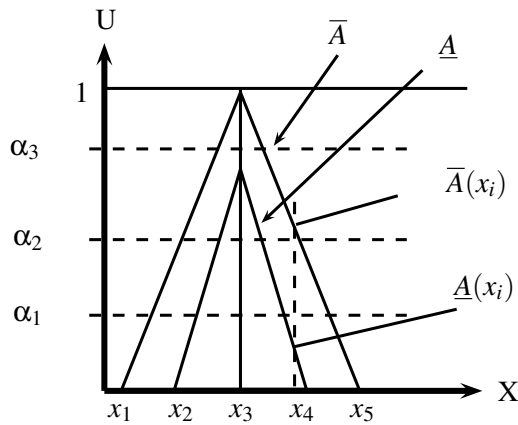


Fig. 4.2. IVFS  $\hat{A}$

interval. This gives an insight to the uncertainty in the non-specificity measure itself, it is not

precise. A single number may be needed in some situations, therefore, either an average non-specificity or a summation can be used. The average non-specificity is defined in equation (4.23) and a cumulative non-specificity is defined in equation (4.24).

$$ns_{av}(\hat{A}) = \frac{ns(\underline{A}) + ns(\overline{A})}{2} \quad (4.23)$$

$$ns_{sum}(\hat{A}) = ns(\underline{A}) + ns(\overline{A}) \quad (4.24)$$

The summation is more intuitive in the sense that there is non-specificity in each of the upper and lower memberships and an average will be less non-specific than the upper membership function. The summation on the other hand shows clearly that non-specificity is an accumulation of the non-specificity of both membership functions.

#### 4.4.3 Non-specificity for T2FSs

Now an extension of the axiomatic definition of non-specificity to T2FSs is provided. A non-specificity measure for a T2FS should satisfy the following:

- **(NS1):**  $ns(\tilde{A}) \in [0, \infty+)$
- **(NS2):**  $\tilde{A} = \{x\} \Rightarrow ns(\tilde{A}) = 0$
- **(NS3):**  $\tilde{A} \subseteq \tilde{B} \Rightarrow ns(\tilde{A}) \leq ns(\tilde{B})$

Again these axioms are direct extensions of those for FSs and carry the same semantics. It is important to stress on the fact that the only specific situation is the singleton. Now the non-specificity for T2FSs can be defined using the concept of an  $\alpha$ -plane.

**Theorem 4.4.2** *Given a T2FS,  $\tilde{A} \in \tilde{F}(X)$ , then the union of all the non-specificity measures of its decomposed  $\alpha$ -planes is a non-specificity measure for T2FSs, i.e.,*

$$ns(\tilde{A}) = \bigcup_{\forall \tilde{\alpha}} \tilde{\alpha} ns(\tilde{A}_{\tilde{\alpha}}) \quad (4.25)$$

**Proof.** Check if the function (measure) satisfies the axioms as follows:

Axioms NS1 and NS2:

Since  $\tilde{A}_{\tilde{\alpha}}$  is an IVFS, then from (NS1) of IVFSs  $ns(\tilde{A}_{\tilde{\alpha}}) \in [0, \infty+)$  then

$$\bigcup_{\forall \tilde{\alpha}} \tilde{\alpha} ns(\tilde{A}_{\tilde{\alpha}}) \in [0, \infty+)$$

and from (NS2) of IVFSs

$$\tilde{A}_{\tilde{\alpha}} = \{x\} \Rightarrow ns(\tilde{A}_{\tilde{\alpha}}) = 0$$

hence,

$$\bigcup_{\forall \tilde{\alpha}} \tilde{\alpha} ns(\tilde{A}_{\tilde{\alpha}}) = \{x\}$$

The proof of axiom NS3 follows from NS3. ■

Considering the outcome of this function, based on the non-specificity measure of the decomposed  $\alpha$ -planes the result may be a distribution of intervals, a distribution of numbers, or even a single number if an average or a summation is used. The average non-specificity can be defined as follows:

$$ns_{av}(\tilde{A}) = \frac{1}{n} \sum_{i=1}^n ns_{av}(\tilde{A}_{\tilde{\alpha}_i}) \quad (4.26)$$

where  $n$  is the number of  $\alpha$ -planes. The summation non-specificity can be defined as follows:

$$ns_{sum}(\tilde{A}) = \sum_{i=1}^n ns_{sum}(\tilde{A}_{\tilde{\alpha}_i}) \quad (4.27)$$

this function is mostly useful when comparing different types of fuzzy sets as will be shown later in this section. Here the importance of the measure of non-specificity and the  $\alpha$ -plane representation of fuzzy sets in understanding T2FSs is emphasised. In a discussion by Niewiadomski (2007), a subjective comparison between IVFS and IT2FSs is formulated. It is well known that IT2FSs equate to IVFSs at a set definition level (Niewiadomski 2007, Mendel et al. 2006, Mendel 2007, Wu & Mendel 2007b). Some arguments in (Niewiadomski 2007) suggest that in some situations IVFSs are not equal to IT2FSs. An underlying fault in the arguments presented in that paper hindered further investigation on the suggestion. The fault occurred on the event of defining measures that has different interpretations for each model. Moreover, no single measure or function that span across these fuzzy sets has been used to support such claims. Now the relationship between IT2FSs, IVFSs and T2FSs is discussed using non-specificity measures. It is believed that in certain circumstances the non-specificity measure supports Niewiadomski's arguments and substantiates them with a measure which has the same interpretation across all representations. First, as stated in (Mendel et al. 2006) "*the FOU is a complete description of an IT2FS*", this is due to the absence of new information conveyed by the secondary grades as they are all equal and at unity. Geometrically, observe that the FOU is a 2D plane and IT2FSs are 3D shapes as shown in figure (4.5). The non-specificity by definition, measures how an observation or a statement is specific e.g. "*John's age is between 30 and 35*" is crisp, but not specific. This set is mathematically represented as follows  $A^1 = [30, 35]$ . The only specific statement is the singleton e.g. "*John's age is 32*" represented as  $A^2 = \{32\}$ . Let us consider the following triangular normal fuzzy set that represents "*John's age is about 32*",  $A^3 = \langle 30, 32, 35 \rangle$ . This set is fuzzy and non-specific. A comparison between the three sets is found in figure (4.3). The non-specificity for the three sets are shown in table (4.1). Observe that the non-specificity of the crisp interval is greater than

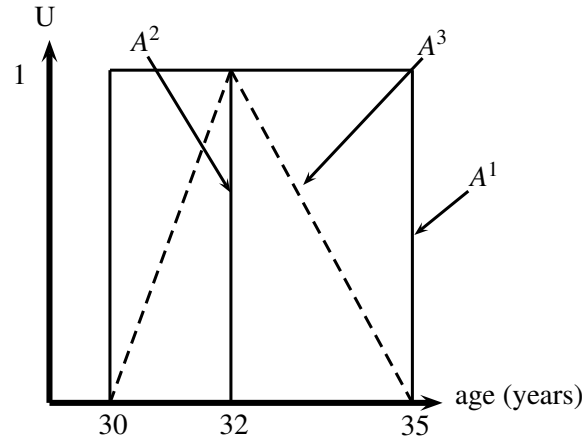


Fig. 4.3. John's age example where  $A^1$  is a crisp interval,  $A^2$  is a singleton and  $A^3$  is a FS.

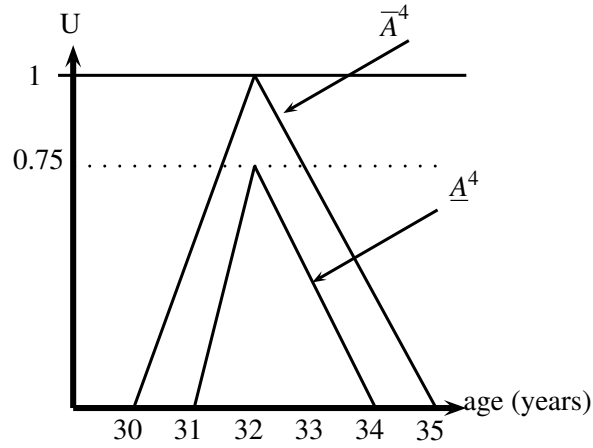


Fig. 4.4. John's age example where  $\hat{A}^4$  is an IVFS.

the non-specificity of the fuzzy set. It is also apparent that the crisp interval has the maximum non-specificity over all the normal fuzzy sets defined within the boundaries of the same support and core, i.e.,

$$\frac{1}{h_A} \sum_{\forall \alpha} \log_2 (1 + \|A_\alpha\|) \leq \frac{1}{h_A} \sum_{\forall \alpha} \log_2 (1 + \|A_0\|) \quad (4.28)$$

Semantically, when informed by  $A^2$ , the perception one conceives of the age of John is equally distributed between 30 and 35, and it can be anywhere within this interval. On the other hand, when informed by  $A^3$ , the perception of the age of John is more specific. Of course it depends totally on the perception of the word "about". Now, define an IVFS  $\hat{A}^4 = \langle 30, 32, 32, 35, 31, 32, 32, 33, 0.75 \rangle$  constructed from a host of FSs representing different people's opinions about John's age (e.g. us-

ing the Interval Approach in Liu & Mendel (2008)) as seen in Figure 4.4. In table (4.1), an interval non-specificity measure is used based on equation (4.20). This shows that the non-specificity itself is non-specific. This appears to be analogous to the term, *higher order vagueness*, used by some philosophers ((Williamson 1996, Keefe & Smith 1999)). The question asked here is whether the non-specificity in this case is an accumulation, an average, or can be interpreted in another way. The interval non-specificity measure can be interpreted as an accumulation of the two end points of the interval it will ensure that this measure is greater than the maximum non-specificity of any embedded FS defined within the boundaries of this set. An average will not have this property, indeed an average will always be less than the maximum non-specificity of any of these embedded sets. Viewing the complete IT2FS in Figures 4.5 and Figure 4.6, their vertical slices are intervals meaning that they are the most non-specific sets within their supports. When looking at an IT2FS from a different view, as a special case of a T2FS, it is actually the most non-specific case of a T2FS within its support. Mathematically, this fact can be described as follows

$$\bigcup_{\forall \tilde{\alpha}} \tilde{\alpha}ns(\tilde{A}_{\tilde{\alpha}}) \leq \bigcup_{\forall \tilde{\alpha}} \tilde{\alpha}ns(\tilde{A}_{\tilde{0}}) \quad (4.29)$$

Clearly, the amount of information within the IT2FS is the minimum unless all the rest of  $\alpha$ -planes are considered negligible and only the FOU is considered for calculation. To summarise, if the third dimension is negligible and in some applications it is, an IT2FS should be considered as an IVFS and all the results obtained in the literature for IT2FS can be considered for IVFS and vice versa. Consequently, new semantics for constructing the IT2FS that make use of this third dimension should be utilised. Wu & Mendel (2009) compared different uncertainty measures

Table 4.1. John's age non-specificity analysis

	<i>Bits</i>
$ns(A^1)$	2.58
$ns(A^2)$	0
$ns(A^3)$	1.72
$ns(A^4)$	[0.96, 1.72]

for IVFSs and concluded that “*Cardinality is the most representative uncertainty measure for an IVFS: its centre is representative intra-personal uncertainty measure, and its length is a representative inter-personal uncertainty measure*”. In the following the correlation between the cardinality and non-specificity using the same dataset presented in (Wu & Mendel 2008a) is discussed. The same cardinality measures are used which are given in equations 4.30 and 4.32.

$$card_{av}(\tilde{A}) = \frac{card_n(\tilde{A}_e) + \overline{card_n}(\tilde{A}_e)}{2} \quad (4.30)$$



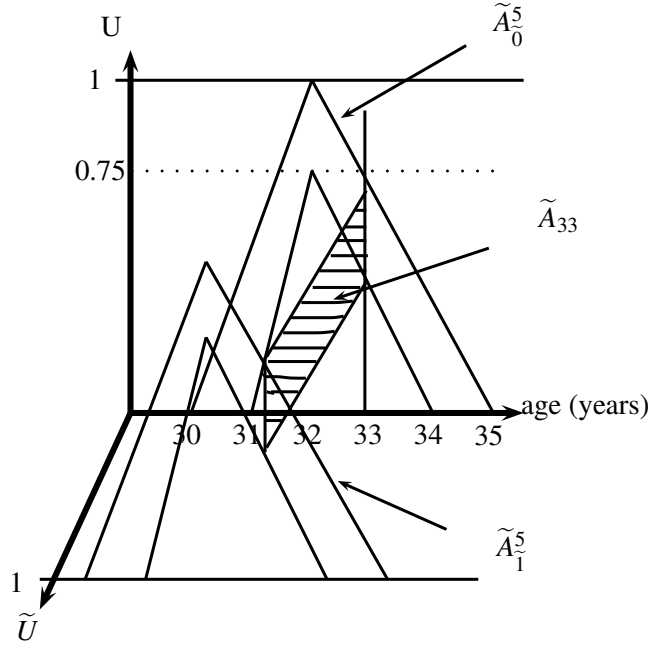


Fig. 4.5. John's age example where  $\tilde{A}^5$  is an IT2FS, and  $\tilde{A}_{33}$  is the vertical slice at 33.

and

$$card_{len}(\tilde{A}) = \overline{card}_n(\tilde{A}_e) - \underline{card}_n(\tilde{A}_e) \quad (4.31)$$

where

$$card_n(A) = \frac{|X|}{N} \sum_{i=1}^N \mu_A(x_i) \quad (4.32)$$

Two proposed measures of non-specificity for IVFSs are used  $ns_{av}(\hat{A})$  and  $ns_{sum}(\hat{A})$ . Taking in consideration that non-specificity measures are cardinality-based measures, they are expected to result in high correlation. The correlation defined in equation (4.33) is used.

$$corr(f_1, f_2) = \frac{\sum_{i=1}^n f_1(\hat{A}_i) f_2(\hat{A}_i)}{\sqrt{\sum_{i=1}^n f_1^2(\hat{A}_i) \sum_{i=1}^n f_2^2(\hat{A}_i)}} \quad (4.33)$$

where  $n$  is the number of entries, which is 32 words in the data set taken from (Wu & Mendel 2008a). Table 4.2 shows the results, that shows a strong correlation of (1.00) between both average and sum non-specificity and average cardinality, and a (0.96) correlation between the length of the cardinality and both these measures. Based on these results it is clear that cardinality and non-specificity are good representatives of each other and can be used to measure intra-personal uncertainty. Fuzziness is another way of measuring intra-personal uncertainty, its axioms and properties are studied extensively in the literature. In their comparative study Wu and Mendel

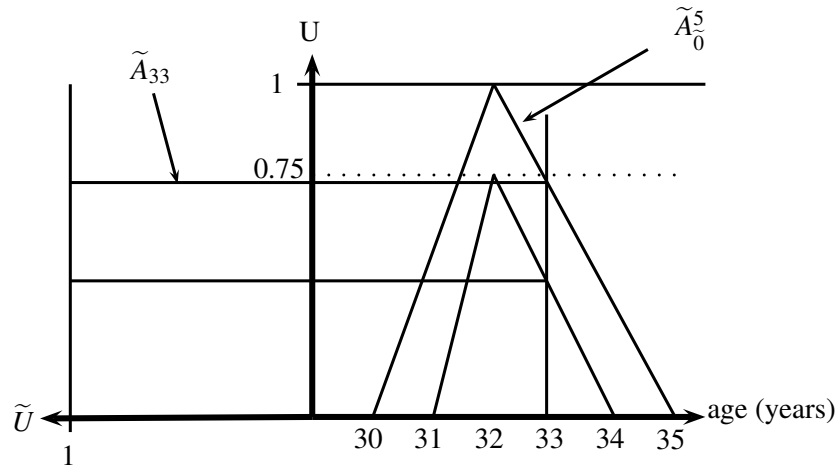


Fig. 4.6. 2D representation of John's age example in figure (4.5) where  $\tilde{A}^5$  is an IT2FS, and  $\tilde{A}_{33}$  is the vertical slice at 33.

found that the cardinality is a good representative of fuzziness, however, by definition cardinality and fuzziness are different kinds of uncertainty, and the same applies to non-specificity. It is clear that a set could have low fuzziness or even zero fuzziness (crisp) and still have high non-specificity and indeed high cardinality, this is clear in John's age example. Fuzziness and imprecision are two quite different kinds of uncertainty, where fuzziness examines the extent of distinction between the crisp and fuzzy sets, the non-specificity examines the amount of imprecision or information contained in a set whether it is fuzzy or crisp (Klir & Folger 1988, Klir 2006). In the end of this

Table 4.2. Correlation between non-specificity and cardinality measures of IT2FS

	$ns_{av}(\hat{A})$	$ns_{sum}(\hat{A})$
$card_{av}(\hat{A})$	1.00	1.00
$card_{len}(\hat{A})$	0.96	0.96

investigation the simplicity of defining uncertainty measures lies on the procedure for defining these operations. Two distinct but related procedures are presented, one that relies on the  $\alpha$ -plane RT through the novel  $\alpha$ -plane EP presented in Chapter 3. The other relies on the novel  $\alpha$ -cut RT through the  $\alpha$ -cut EP, both presented in Chapter 3 for the first time. The first procedure can be summarised as follows:

1. Find if the function is cutworthy.
2. Extend the function (and/or axioms) to IVFSs (The literature of IVFSs is rich with functions an operations).
3. Determine the  $\alpha$ -planes.

4. Use the function in step (2) with each  $\alpha$ -plane.
5. Use any averaging function to produce a scalar result if required.

The second procedure, does not differ greatly and can be summarised as follows:

1. Find if the function is cutworthy.
2. Extend the function (and/or axioms) to T2FSs, directly.
3. Determine the  $\alpha$ -cuts.
4. Use the function in step (2) with each  $\alpha$ -cut.
5. Use any averaging function to produce a scalar result if required.

## 4.5 Summary

In this chapter an investigation is formulated on some uncertainty measures including cardinality, similarity, subsethood, fuzziness and non-specificity for type-2 fuzzy sets. The question of how can the  $\alpha$ -EP based on  $\alpha$ -planes and  $\alpha$ -cuts be used to define operations in a simple and straight forward manner is discussed. In particular, decomposing T2FSs into  $\alpha$ -cuts provides a very simple way of looking at these T2FSs. The main argument of this chapter is that the mathematical formalism presented in Chapter 3, i.e., the  $\alpha$ -cut representation theorem along with the calculus for manipulating this formalism, i.e., the  $\alpha$ -cut extension principle are suitable for defining measures of uncertainty for T2FSs. This has been shown to be true by the five concepts defined in this chapter and listed in Table 4.3. The other objective is to provide evidence of the elegant nature of both  $\alpha$ -based extension principles (i.e.  $\alpha$ -plane and  $\alpha$ -cut), that can be seen in simple and straight forward manner in which functions, operations and even axioms are extended. The measures chosen in this chapter are used in many applications, and the ability to define many other measures are straight forward following the procedures presented above. The next step would be to use these measures in applications, which is out of the thesis scope. The applications of the uncertainty measures presented in this chapter are already well documented in the literature of fuzzy sets. To summarise, the following table shows the contributions presented in this chapter:

Table 4.3. The contributions presented in this chapter.

Concept	Contribution	Reference
Cardinality of T2FSs using $\alpha$ -planes	New alternative	Theorem 4.1.1
Similarity of T2FSs using $\alpha$ -planes	New alternative	Theorem 4.2.1
Subsethood of T2FSs using $\alpha$ -planes	Novel	Theorem 4.2.2
Fuzziness of T2FSs using $\alpha$ -planes	Novel	Eq. 4.8
Non-specificity of IVFSs using $\alpha$ -planes	Novel	Definition 4.4.1
Non-specificity of T2FSs using $\alpha$ -planes	Novel	Definition 4.4.2

## Chapter 5

# Type-2 Fuzzy Numbers and Arithmetic

In this chapter type-2 fuzzy numbers are defined and some basic arithmetic operations are investigated. Type-2 fuzzy numbers are the first step in the definition of type-2 fuzzy mathematics. One can never imagine solving for example type-2 fuzzy equations without being able to define a type-2 fuzzy number. The uncertainty layer surrounding a real number is extended in the same pattern followed throughout this thesis. The hierarchy of uncertain numbers start with the definition of an interval as a first conception about the uncertainty about a number, then fuzzy numbers come to add extra layer of uncertainty about the value of the number. Interval valued fuzzy numbers and type-2 fuzzy numbers form higher levels of uncertainty about a number. In fact it is a number which can be described linguistically. Many applications in the literature of standard fuzzy numbers are used (Martin & Klir 2006, Yager 2004, Wang & Li 1998). In reality all the control applications of standard fuzzy sets are fuzzy numbers. Type-2 fuzzy numbers and their associated arithmetic operations have not received the same attention, only two main contributions appear in the literature that apply arithmetic operations. One makes use of the wavy slice representation theorem and the other uses the extension principle through the vertical slice representation theorem (Coupland & John 2003, Blewitt et al. 2007). In this chapter  $\alpha$ -cuts for type-2 fuzzy sets are used as the basis for defining and manipulating type-2 fuzzy numbers. The examination of type-2 fuzzy arithmetic through  $\alpha$ -cuts for both discrete and continuous domains are discussed. Also arithmetic operations for interval valued fuzzy sets are considered as a stage before full type-2 fuzzy sets. Quasi type-2 fuzzy numbers first identified by Hamrawi & Coupland (2009b) are further refined. It is considered a step on the journey between interval valued fuzzy numbers and type-2 fuzzy numbers.

### 5.1 Type-2 Fuzzy Numbers

A fuzzy number (FN) is defined to be a fuzzy set that is both normal and convex (Dubois & Prade 1980, Kaufmann & Gupta 1985). Normality is required in order to capture the concept

of a number within the framework of fuzzy sets. A fuzzy number can be interpreted as a set of real numbers close to a specific crisp number (Klir & Yuan 1995). In other words when all the uncertainty about a fuzzy number disappears, it reduces to a crisp number or in some cases an interval number. Let,  $A$ , be a FS that is subnormal (i.e.  $h(A) \neq 1$ ), then it is evident that the FS will not reduce to a crisp number at all. This class of FSs does not provide any arithmetical meaning. Convexity in the other hand is required to allow meaningful arithmetic operations to be performed on fuzzy sets. For example the well established methods from interval analysis can be utilised by using the  $\alpha$ -cut RT and requiring the FSs to be continuous (Klir & Yuan 1995). It is also apparent that a FN which is non-convex have not been used in the literature, it is actually very hard to find an account of non-convex FSs useful in applications despite some plausible arguments by Garibaldi & John (2003), and Garibaldi et al. (2004). The formal definition of a fuzzy number is as follows:

**Definition 5.1.1 (Fuzzy Number)** *A fuzzy set is said to be a fuzzy number if and only if, it is normal and convex.*

*Adapted from Klir & Yuan (1995)*

The importance of being normal is illustrated by the fact that  $\exists x_0 \in A$  such that  $A(x_0) = 1$ . When uncertainty disappears there will exist at least this  $x_0$  to represent a crisp number. It might even be convenient to restrict a fuzzy number to have only one element of the domain that has a membership grade at unity. In this case intervals are not considered as uncertain crisp numbers, but different entities and their extension to fuzzy theory is called fuzzy intervals (Martin & Klir 2006). Mathematically, this distinction is not very significant as both situations are subsumed in the FNs definition, thus the FN definition is maintained to represent fuzzy numbers or fuzzy intervals. Whenever a FS is not normal it is called *subnormal*, whenever a FS is subnormal and convex it said to be called a *sub-number*.

**Definition 5.1.2 (Fuzzy sub-number)** *A fuzzy set is said to be a fuzzy sub-number if and only if, it is subnormal and convex.*

The convex nature of FNs or FsNs can be described using the  $\alpha$ -cut RT by requiring that all the  $\alpha$ -cuts be convex sets, i.e.,  $\forall \alpha, A_\alpha$  is convex. Coupland & John (2003) defined a type-2 fuzzy number (T2FN) to be a type-2 fuzzy set having a numerical domain. There were no assumption about normality defined for T2FNs at that point. Despite this fact, the examples used by Coupland & John (2003) assume having a domain value that has a primary membership grade at unity associated with a secondary grade at unity. This is in-effect an assumption of normality, see Definition 3.4.10. In the framework of IVFSs, interval-valued fuzzy numbers (IVFNs) are defined to be convex and normal (Hong & Lee 2002, Wang & Li 1998). The following definition of IVFNs is defined by Hong & Lee (2002).

**Definition 5.1.3** An IVFS,  $\hat{A} = (\underline{A}, \bar{A})$ , is said to be an interval valued fuzzy number if and only if,

1.  $\hat{A}$  is normal, i.e.,  $\exists x_0 \in \hat{A}$  such that  $\hat{A}(x_0) = [1, 1]$ , in other words  $\underline{A}$  and  $\bar{A}$  are both normal.
2.  $\hat{A}$  is convex, i.e.,  $\underline{A}$  and  $\bar{A}$  are both convex.

*Adapted from Hong & Lee (2002)*

This definition means that  $\hat{A}$  is an IVFN if and only if  $\underline{A}$  and  $\bar{A}$  are both fuzzy numbers. This definition is intuitive but somewhat restrictive. Requiring both the LMF and UMF to be FNs eliminates a wide range of sets from being considered as IVFNs. A more general definition of the normality property is desired. A class of normal IVFSs are implied by the work of Kaufmann & Gupta (1985) in which the LMFs are found to be in some examples subnormal. In this situation the uncertainty about the crisp number deludes when uncertainties about both the LMF and UMF disappear, and hence it generalises the concept of normal FSs. Note that there is a point in which a crisp number can be achieved that depend on the UMF and LMF. To explain this point, let  $\hat{A}$  be a normal IVFN defined on the domain of real numbers. It means that in order for a crisp number to be represented using an IVFN then,

$$\hat{A} \in \mathcal{C}(X) \Leftrightarrow \hat{A}(x) = [1, 1], x \in X$$

which translates to  $\underline{A}(x) = 1$  and  $\bar{A}(x) = 1$ . A less restrictive definition is needed that make use of the second normality definition, i.e., the normal IVFS of Definition 3.4.9. This less restrictive condition is widely used in applications (e.g. computing with words (Mendel & Wu 2008)), and at the same time it is a generalisation of the perfectly normal IVFS as explained earlier in Definition 3.4.8. The two situations are distinguished in order to make it simple to communicate about each situation. Then IVFNs can be defined utilising the concept of normality and convexity.

**Definition 5.1.4 (IVFN)** An IVFS is said to be an interval valued fuzzy number (IVFN) if,

- it is convex, i.e., its LMF and UMF are both convex.
- it is normal, i.e., its UMF is normal.

Based on the definition of normal IVFSs, a special class of IVFNs can be described. It is based on the perfectly normal IVFS definition. This special is called perfect IVFN, and these IVFNs can be described using FNs as follows:

**Definition 5.1.5** An IVFN can be described as having a UMF which is a FN, and LMF which is a FsN as described in Definition 5.1.2. A special case called perfect IVFN occurs when both its LMF and UMF are FNs, in this case it is called a perfect IVFN.

This classification helps communicate about different situations where an IVFS is considered an IVFN. Earlier in Figure 3.11 both sets are IVFNs, in fact the perfectly normal IVFS is a perfect IVFN and the normal IVFS is an IVFN. Another less important classification are the class of IVFSs that reduce into a FsN, i.e. the condition in which the LMF and UMF are both FsNs.

**Definition 5.1.6 (IVFsN)** *An IVFS defined on real numbers is called an interval valued fuzzy sub-number (IVFsN) if both its LMF and UMF are fuzzy sub-numbers.*

From these two definitions it is obvious that the classification of IVFNs is determined by the IVFS membership grades represented by the UMF and LMF. Note that these definitions are determined using the definition of a fuzzy number rather than using the membership grades. The reason behind this attempt is, first, make use of already available definitions. Second, it gives insight to the reduction rule in which IVFSs are reduced to FSs. On the other hand, the classification of T2FSs is determined by its membership grades represented by the primary and secondary grades. In the same manner one can utilise the already available definitions of IVFNs to define T2FNs.

**Definition 5.1.7 (T2FN)** *A T2FS defined on real numbers is said to be a type-2 fuzzy number (T2FN) if,*

- *it is convex, i.e., its FOU, PS, and all VSs are convex.*
- *it is normal, i.e., its FOU and PS are both normal IVFSs.*

The different classes of normal T2FSs carry over to T2FNs and allow them to have different classes. These classes can also be described in terms of IVFNs.

**Definition 5.1.8 (T2FN)** *A T2FN can be described by having both its FOU and PS as IVFNs. Two special cases can be distinguished,*

- *perfect T2FN, if both its FOU and PS are perfect IVFNs, and*
- *semi-perfect T2FN, if its FOU is an IVFN and its PS is perfect IVFN.*

These conditions reduce to IVFNs, FN, and crisp numbers when uncertainties disappear. Earlier in Figure 3.12 these sets are T2FNs, in fact the perfectly normal T2FSs is a perfect T2FN, the semi-perfectly normal T2FS is a semi-perfect T2FN, and the normal T2FS is a perfect T2FN. The final situation that can be defined is the situation in which the T2FS reduces to an IVFsN or a FsN.

**Definition 5.1.9 (T2FsN)** *A T2FS defined on real numbers is said to be a type-2 fuzzy sub-number (T2FsN) if its FOU its PS are IVFsNs.*



These sub-numbers are sometimes useful to describe convex sets that are used for or result from some operations. In some situations the sub-normality condition can be uplifted to a normal set by applying a function (Klir 2006). Convexity also can be achieved by applying a function that translates any non-convex set to a convex set (e.g. interpolation), such a function can be called a convexifying function. Walker & Walker (2005) and Walker & Walker (2009) applied a convexifying function to their investigation without calling it this name, indeed it is a convexity assumption in an algebraic setting. The normalising and convexifying functions are beyond the scope of investigation of this thesis and considered as a goal for further research. Thus, all sets involved in arithmetic operations are assumed to be convex and normal. The other important consideration of all the types of fuzzy sets involved in this investigation is continuous vs discrete sets. In continuous domains a FN can be defined using piecewise functions which can be linear or non-linear.

**Definition 5.1.10** *Let,  $A$ , be a FS with height,  $h(A) = h_A$ , defined on real numbers. If there exists a closed interval,  $[m_1, m_2] \neq \emptyset$ , such that,*

$$A(x) = \begin{cases} h_A & x \in [m_1, m_2] \\ l(x) & x \in [s, m_1) \\ r(x) & x \in (m_2, e] \\ 0 & x \in (-\infty, s); x \in (e, \infty) \end{cases} \quad (5.1)$$

where  $l(x) \in [0, 1]$  is monotonically increasing and continuous from the right, and  $r(x) \in [0, 1]$  is monotonically decreasing and continuous from the left. Then  $A$  is a FN if  $h_A = 1$  and a FsN if  $h_A < 1$ .

*Adapted from Klir & Yuan (1995)*

If,  $l(x)$  and  $r(x)$ , are linear functions then,  $A$ , is trapezoidal and can be totally represented by the quintuple,  $\langle s, m_1, m_2, e, h_A \rangle$ , furthermore, if,  $m_1 = m_2 = m$ , then,  $A$ , is triangular and can be fully represented by the quadruple,  $\langle s, m, e, h_A \rangle$ . If  $A$  is a FN then  $h_A$  in these parameters can be dropped since it is always at unity. Notice also that if,  $A$ , is a FN defined on a continuous domain, then it can be represented as a set of nested intervals using their  $\alpha$ -cuts, i.e.,

$$\begin{aligned} A &= \bigcup_{\forall \alpha} \alpha [{}^L a_\alpha, {}^R a_\alpha] \\ &= \bigcup_{\forall \alpha} \alpha [l^{-1}(\alpha), r^{-1}(\alpha)] \end{aligned} \quad (5.2)$$

where,  $A_\alpha = [{}^L a_\alpha, {}^R a_\alpha]$ , is the  $\alpha$ -cut of FN,  $A$ , and  $l^{-1}$  and  $r^{-1}$  are inverse functions. For examples, let  $A = \langle s, m_1, m_2, e, h_A \rangle$  be a FN with linear functions applied to Eq. 5.1. Then, the linear functions

are defined as:

$$l(x) = \frac{h_A(x-s)}{m_1-s} \text{ and } r(x) = \frac{h_A(e-x)}{e-m_2} \quad (5.3)$$

It is clear since  $A$  is a FN then  $h_A = 1$  and the  $\alpha$ -cuts may be used:

$$A_\alpha = \left[ s + \frac{(m_1-s)\alpha}{h_A}, e - \frac{(e-m_2)\alpha}{h_A} \right] \quad (5.4)$$

All these concepts are extended in order to define IVFNs, since these IVFNs are represented by their UMFs which are FNs, and LMFs which are either FNs or FsNs. Let,  $\hat{A} = (\underline{A}, \bar{A})$ , represent a perfect IVFN defined by its LMF and UMF. Using Eq. 5.2 individually for each of the LMF and UMF result in the following equation for the LMF:

$$\begin{aligned} \underline{A} &= \bigcup_{\forall \alpha} \alpha [{}^L \underline{a}_\alpha, {}^R \underline{a}_\alpha] \\ &= \bigcup_{\forall \alpha} \alpha [{}^L \underline{a}_\alpha^{-1}(\alpha), {}^R \underline{a}_\alpha^{-1}(\alpha)] \end{aligned}$$

,and the following equation for the UMF:

$$\begin{aligned} \bar{A} &= \bigcup_{\forall \alpha} \alpha [{}^L \bar{a}_\alpha, {}^R \bar{a}_\alpha] \\ &= \bigcup_{\forall \alpha} \alpha [{}^L \bar{a}_\alpha^{-1}(\alpha), {}^R \bar{a}_\alpha^{-1}(\alpha)] \end{aligned}$$

combining the LMF and UMF under the same  $\alpha$ -levels result in the  $\alpha$ -cut representation of IVFSs in Definition 3.1.1, i.e.,  $\hat{A}_\alpha = (\underline{A}_\alpha, \bar{A}_\alpha)$ . The following representation of IVFNs uses the  $\alpha$ -cut intervals:

$$\begin{aligned} \hat{A} &= \bigcup_{\forall \alpha} \alpha (\underline{A}_\alpha, \bar{A}_\alpha) \\ &= \bigcup_{\forall \alpha} \alpha ([{}^L \underline{a}_\alpha, {}^R \underline{a}_\alpha], [{}^L \bar{a}_\alpha, {}^R \bar{a}_\alpha]) \end{aligned} \quad (5.5)$$

If the value of  $\alpha$  is greater than the hight of the LMF,  $h(\underline{A})$ , then the  $\alpha$ -cuts of the LMFs are empty sets. In order to reflect the different kinds of IVFNs, the  $\alpha$ -cut of an IVFN can be represented as shown in Eq. 3.5.

$$\hat{A}_\alpha = \left( \begin{cases} [{}^L \underline{a}_\alpha, {}^R \underline{a}_\alpha], [{}^L \bar{a}_\alpha, {}^R \bar{a}_\alpha] & , \alpha \leq h(\underline{A}) \\ \emptyset, [{}^L \bar{a}_\alpha, {}^R \bar{a}_\alpha] & , \alpha > h(\underline{A}) \end{cases} \right) \quad (5.6)$$

The two situations can be formulated using this equation. For example if  $\hat{A}$  is a perfect IVFN then the  $\alpha$ -cut of the LMF will not be an empty set at all. The next step is to define T2FNs using these

methods. The idea is to use the  $\alpha$ -cut RT of T2FSs, i.e.,

$$\tilde{A} = \bigcup_{\forall \tilde{\alpha}} \tilde{\alpha} \left( \bigcup_{\forall \alpha} \alpha \tilde{A}_{\tilde{\alpha}, \alpha} \right)$$

where,  $\tilde{A}_{\tilde{\alpha}, \alpha} = (\underline{A}_{\tilde{\alpha}, \alpha}, \overline{A}_{\tilde{\alpha}, \alpha})$ . To explain,  $\underline{A}_{\tilde{\alpha}, \alpha}$  represents the  $\alpha$ -cut of LMF,  $\underline{A}_{\tilde{\alpha}}$ , of  $\alpha$ -plane,  $\tilde{A}_{\tilde{\alpha}}$ . Also,  $\overline{A}_{\tilde{\alpha}, \alpha}$  represents the  $\alpha$ -cut of UMF,  $\overline{A}_{\tilde{\alpha}}$ , of  $\alpha$ -plane,  $\tilde{A}_{\tilde{\alpha}}$ . If,  $\tilde{A}$ , is defined on a continuous domain then the  $\alpha$ -cut of any LMF of any  $\alpha$ -plane can be described using an interval as follows:

$$\underline{A}_{\tilde{\alpha}, \alpha} = [{}^L \underline{a}_{\tilde{\alpha}, \alpha}, {}^R \underline{a}_{\tilde{\alpha}, \alpha}]$$

, and the  $\alpha$ -cut of any UMF can be described as follows:

$$\overline{A}_{\tilde{\alpha}, \alpha} = [{}^L \overline{a}_{\tilde{\alpha}, \alpha}, {}^R \overline{a}_{\tilde{\alpha}, \alpha}]$$

Then a T2FN can be described using the collection of all these intervals using the  $\alpha$ -cut representation theorem.

$$\tilde{A}_{\tilde{\alpha}, \alpha} = \left( \begin{cases} [{}^L \underline{a}_{\tilde{\alpha}, \alpha}, {}^R \underline{a}_{\tilde{\alpha}, \alpha}], [{}^L \overline{a}_{\tilde{\alpha}, \alpha}, {}^R \overline{a}_{\tilde{\alpha}, \alpha}] & , \alpha \leq h(\underline{A}_{\tilde{\alpha}}) \\ \emptyset, [{}^L \overline{a}_{\tilde{\alpha}, \alpha}, {}^R \overline{a}_{\tilde{\alpha}, \alpha}] & , \alpha > h(\underline{A}_{\tilde{\alpha}}) \end{cases} \right) \quad (5.7)$$

where,  $h(\underline{A}_{\tilde{\alpha}})$ , is the height of the LMF,  $\underline{A}_{\tilde{\alpha}}$ , of  $\alpha$ -plane,  $\tilde{A}_{\tilde{\alpha}}$ . In some applications one may need to restrict a fuzzy set to a specific form, e.g. in computing with words Martin & Klir (2006) and Klir & Sentz (2006) defined a procedure to convert any given convex fuzzy set to a fuzzy interval which is expressed in some standard form using some specific criteria. Another special form is discussed by Mendel & Wu (2007) in their perceptual reasoning framework, a restricted IVFS is used as a model for a word. Some observations about the shape of the centroid led Mendel & Liu (2008) to the proposition of a quasi type-2 fuzzy logic system, in which the output of the reasoning stage of the system is restricted to a T2FS represented only by its FOU and PS. Starczewski (2009) in the other hand, also defined a triangular type-2 fuzzy systems in which the result of the reasoning stage is restricted to three FSs. Hamrawi & Coupland (2009b) defined a quasi T2FN (QT2FN), and in this thesis more elaboration and investigation into QT2FNs are conducted. First, Let  $\tilde{A}$  be a T2FS satisfying the following propositions:

P1 All the VSs of the T2FS are FNs, i.e.  $\forall x h(\tilde{A}_x) = 1$ .

P2 All the VSs of the T2FS are piecewise functions of the same type (e.g. linear).

The first proposition assures that the T2FS contains an FOU and a PS. This fact is clear since all the VSs are normal which makes it clear that for all the domain values there is at least one primary

grade with secondary grade at unity. The second property assures that that only a set parameters are needed to define the T2FS which is directly related to the FOU and the PS. These propositions allow a T2FS be completely determined using its FOU and PS.

**Definition 5.1.11 (QT2FS)** *A T2FS is called a quasi T2FS (QT2FS) if it can be completely determined using its FOU and PS.*

What this definition means is that using some simple of parameters a T2FS can be defined. Although it is a restricted form of a T2FS, but useful in some applications that require these restricted forms in order to consolidate for computational complexity. It is a 3D restriction resembling the FS definition of fuzzy intervals completely determined by its core and support. Now, this QT2FS can be restricted to be a special T2FN if both the FOU and PS are restricted to be IVFNs.

**Definition 5.1.12 (QT2FN)** *A T2FN is called a quasi type-2 fuzzy number (QT2FN) if it is completely determined by its FOU and PS.*

QT2FNs satisfy the same conditions required to be T2FNs with the extra restriction of propositions P1 and P2. One of the most popular sets are triangular T2FSs, which can be described by linear piecewise functions representing the vertical slices and both FOU and PS are triangular IVFNs.

## 5.2 Type-2 Fuzzy Arithmetic

In Kaufmann & Gupta (1985), a comprehensive discussion on fuzzy numbers and arithmetic operations is formulated. Most of the sets used in the investigation are continuous fuzzy numbers, and the arithmetic operations developed mostly utilise the  $\alpha$ -cut RT. Because these  $\alpha$ -cuts are assumed to be intervals, interval arithmetic is used to define these operations for FSs. First, let  $[^La, ^Ra]$  and  $[^Lb, ^Rb]$  be two interval numbers, then according to Moore et al. (2009) the arithmetic operations on intervals are described by the following equation:

$$\begin{aligned} [^La, ^Ra] \circ [^Lb, ^Rb] = \\ [\min(^La \circ ^Lb, ^La \circ ^Rb, ^Ra \circ ^Lb, ^Ra \circ ^Rb), \max(^La \circ ^Lb, ^La \circ ^Rb, ^Ra \circ ^Lb, ^Ra \circ ^Rb)] \end{aligned} \quad (5.8)$$

where  $\circ \in \{+, -, \times, \text{div}\}$  and  $0 \notin B$  if  $\circ = \text{div}$ . Second, let

$$A = \bigcup_{\forall \alpha} \alpha [^La_{\alpha}, ^Ra_{\alpha}] , \text{ and } B = \bigcup_{\forall \alpha} \alpha [^Lb_{\alpha}, ^Rb_{\alpha}]$$

be two FNs represented in the interval  $\alpha$ -cut form. Then according to Kaufmann & Gupta (1985), interval operations are extended to FNs as follows:

$$A \circ B = \bigcup_{\forall \alpha} \alpha ([^L a_\alpha, ^R a_\alpha] \circ [^L a_\alpha, ^R a_\alpha]) \quad (5.9)$$

where  $\circ = \{+, -, \times, \text{div}\}$ . It is clear that FN arithmetic is calculated directly using interval arithmetic. These arithmetic operations can be extended from FNs to IVFNs using the IVFS  $\alpha$ -EP of Theorem 3.1.2. For clarity the IVFS  $\alpha$ -EP is customised for binary arithmetic operations, then it can easily be extended to any number of sets. Let,  $\hat{A} = (\underline{A}, \bar{A})$  and  $\hat{B} = (\underline{B}, \bar{B})$  be two IVFNs then:

$$\begin{aligned} \hat{A} \circ \hat{B} &= \bigcup_{\forall \alpha} \alpha (\hat{A}_\alpha \circ \hat{B}_\alpha) \\ &= \bigcup_{\forall \alpha} \alpha ((\underline{A}_\alpha, \bar{A}_\alpha) \circ (\underline{B}_\alpha, \bar{B}_\alpha)) \\ &= \bigcup_{\forall \alpha} \alpha (\underline{A}_\alpha \circ \underline{B}_\alpha, \bar{A}_\alpha \circ \bar{B}_\alpha) \end{aligned} \quad (5.10)$$

This is generalised form and can equally be used for continuous and discrete domains. Now if the domain is continuous, and at the same time be able to directly extend interval arithmetic to IVFNs, then using Eq. 5.5 let the continuous IVFN,  $\hat{A}$  be represented by the interval  $\alpha$ -cuts of the LMF and UMF.

$$\hat{A} = \bigcup_{\forall \alpha} \alpha ([^L \underline{a}_\alpha, ^R \underline{a}_\alpha], [^L \bar{a}_\alpha, ^R \bar{a}_\alpha])$$

Let also the second IVFN,  $\hat{B}$ , be represented in the same manner, i.e.,

$$\hat{B} = \bigcup_{\forall \alpha} \alpha ([^L \underline{b}_\alpha, ^R \underline{b}_\alpha], [^L \bar{b}_\alpha, ^R \bar{b}_\alpha])$$

Interval arithmetic can be extended from FNs by substituting these sets in Eq. 5.10, i.e.,

$$\begin{aligned} \hat{A} \circ \hat{B} &= \bigcup_{\forall \alpha} \alpha (([^L \underline{a}_\alpha, ^R \underline{a}_\alpha], [^L \bar{a}_\alpha, ^R \bar{a}_\alpha]) \circ ([^L \underline{b}_\alpha, ^R \underline{b}_\alpha], [^L \bar{b}_\alpha, ^R \bar{b}_\alpha])) \\ &= \bigcup_{\forall \alpha} \alpha ([^L \underline{a}_\alpha, ^R \underline{a}_\alpha] \circ [^L \underline{b}_\alpha, ^R \underline{b}_\alpha], [^L \bar{a}_\alpha, ^R \bar{a}_\alpha] \circ [^L \bar{b}_\alpha, ^R \bar{b}_\alpha]) \end{aligned} \quad (5.11)$$

This equation works fine with both cases of IVFNs, it is clear for perfect IVFNs because it is a direct application of the definition. In case that either or both IVFNs are not perfect in the sense that either or both have LMFs as FsNs, i.e.,  $h(\underline{A}) \neq 1$  or/and  $h(\underline{B}) \neq 1$ . Then simply the  $\alpha$ -cuts of the LMFs that exceeds its LMF height is an empty set as shown in Eq. 5.6, and the normal crisp

arithmetic that involves empty sets hold. the reason this fact is mentioned and moreover explained in the preceding, is that such IVFNs have caused problems when the  $\alpha$ -cuts were defined differently. Later in this section these problems are highlighted, and for the time being the computation of these sets are explained. Using the same sets  $\hat{A}$  and  $\hat{B}$ , let them be represented using Eq. 5.6 which is more convenient in reflecting any kind of IVFNs.

$$\hat{A} = \bigcup_{\forall \alpha} \alpha \left( \begin{cases} [L\underline{a}_\alpha, R\underline{a}_\alpha], [L\bar{a}_\alpha, R\bar{a}_\alpha] & , \alpha \leq h(\underline{A}) \\ \emptyset, [L\bar{a}_\alpha, R\bar{a}_\alpha] & , \alpha > h(\underline{A}) \end{cases} \right)$$

and also the second IVFN is represented in the same manner, i.e.,

$$\hat{B} = \bigcup_{\forall \alpha} \alpha \left( \begin{cases} [L\underline{b}_\alpha, R\underline{b}_\alpha], [L\bar{b}_\alpha, R\bar{b}_\alpha] & , \alpha \leq h(\underline{B}) \\ \emptyset, [L\bar{b}_\alpha, R\bar{b}_\alpha] & , \alpha > h(\underline{B}) \end{cases} \right)$$

Assume that  $h_{min} = \min(h(\underline{A}), h(\underline{B}))$  and  $h_{max} = \max(h(\underline{A}), h(\underline{B}))$ . Also recall that for any crisp set  $C \in \mathbf{C}(X)$ , the basic arithmetic operations between that crisp set and an empty set result in an empty set, i.e.,  $\emptyset \circ C = \emptyset$ . Also recall that for any interval  $I \in \mathbf{I}(X)$ , the basic arithmetic operations between that interval and an empty set result in an empty set, i.e.,  $\emptyset \circ I = \emptyset$ . With this fact on mind performing arithmetic operations between IVFNs produce the following:

$$\hat{A} \circ \hat{B} = \bigcup_{\forall \alpha} \alpha \left( \begin{cases} [L\underline{a}_\alpha, R\underline{a}_\alpha] \circ [L\underline{b}_\alpha, R\underline{b}_\alpha], [L\bar{a}_\alpha, R\bar{a}_\alpha] \circ [L\bar{b}_\alpha, R\bar{b}_\alpha] & , \alpha \leq h_{min} \\ \emptyset, [L\bar{a}_\alpha, R\bar{a}_\alpha] \circ [L\bar{b}_\alpha, R\bar{b}_\alpha] & , h_{max} \geq \alpha > h_{min} \\ \emptyset, [L\bar{a}_\alpha, R\bar{a}_\alpha] \circ [L\bar{b}_\alpha, R\bar{b}_\alpha] & , \alpha > h_{max} \end{cases} \right)$$

Observe that conditions  $h_{max} \geq \alpha > h_{min}$  and  $\alpha > h_{max}$  have the same results due to the properties of empty sets. Then it can be further reduced to the following form.

$$\hat{A} \circ \hat{B} = \bigcup_{\forall \alpha} \alpha \left( \begin{cases} [L\underline{a}_\alpha, R\underline{a}_\alpha] \circ [L\underline{b}_\alpha, R\underline{b}_\alpha], [L\bar{a}_\alpha, R\bar{a}_\alpha] \circ [L\bar{b}_\alpha, R\bar{b}_\alpha] & , \alpha \leq h_{min} \\ \emptyset, [L\bar{a}_\alpha, R\bar{a}_\alpha] \circ [L\bar{b}_\alpha, R\bar{b}_\alpha] & , \alpha > h_{min} \end{cases} \right) \quad (5.12)$$

The height of the LMF of the result is equal to the minimum height among all LMFs involved. In the view accepted in this thesis, the  $\alpha$ -cuts of IVFSs does not require any extra attention. In contrary, the  $\alpha$ -cuts of continuous IVFNs developed by Kaufmann & Gupta (1985) and widely accepted in the literature is somewhat problematic. These  $\alpha$ -cuts are shown in Eq. 2.16, and applying arithmetic operations to these sets as noted by Wu & Mendel (2008a), "[this method]result in discontinuous or non-convex sets which are neither desirable nor technically correct". Wu & Mendel (2008a) used these sets for aggregation operation using weighted averages, which depends on arithmetic operations. The problems provoked the corrections to their method originally published in Wu & Mendel (2007a). Havens et al. (2010) also used Kaufmann and Gupta's method

to define the Choquet Integral over IVFSs, and faced the same problem not knowing Wu and Mendel's corrections. To solve the problem Havens et al. (2010) restricted their method to perfect IVFNs, although they did not call it this name but it is implied from their restriction. The next step is to extend interval arithmetic to continuous T2FNs with the aid of the T2FS  $\alpha$ -cut EP shown in Theorem 3.3.2. For clarity the T2FS  $\alpha$ -cut EP is customised for binary arithmetic operations, then it can easily be extended to any number of sets. Let,  $\tilde{A} = \bigcup_{\forall \tilde{\alpha}} \tilde{\alpha} \bigcup_{\forall \alpha} \alpha \tilde{A}_{\tilde{\alpha}, \alpha}$  and  $\tilde{B} = \bigcup_{\forall \tilde{\alpha}} \tilde{\alpha} \bigcup_{\forall \alpha} \alpha \tilde{B}_{\tilde{\alpha}, \alpha}$  be two T2FNs represented by the T2FS  $\alpha$ -cut RT. These  $\alpha$ -cuts can be represented using Eq. 5.7, i.e., it breaks down to a pair of distinct crisp sets as follows:

$$\tilde{A}_{\tilde{\alpha}, \alpha} = \left( \begin{cases} [\underline{a}_{\tilde{\alpha}, \alpha}, \underline{a}_{\tilde{\alpha}, \alpha}], [\bar{a}_{\tilde{\alpha}, \alpha}, \bar{a}_{\tilde{\alpha}, \alpha}] & , \alpha \leq h(\underline{A}_{\tilde{\alpha}}) \\ \emptyset, [\bar{a}_{\tilde{\alpha}, \alpha}, \bar{a}_{\tilde{\alpha}, \alpha}] & , \alpha > h(\underline{A}_{\tilde{\alpha}}) \end{cases} \right)$$

and the same applies to T2FN  $\tilde{B}$ , i.e.,

$$\tilde{B}_{\tilde{\alpha}, \alpha} = \left( \begin{cases} [\underline{b}_{\tilde{\alpha}, \alpha}, \underline{b}_{\tilde{\alpha}, \alpha}], [\bar{b}_{\tilde{\alpha}, \alpha}, \bar{b}_{\tilde{\alpha}, \alpha}] & , \alpha \leq h(\underline{B}_{\tilde{\alpha}}) \\ \emptyset, [\bar{b}_{\tilde{\alpha}, \alpha}, \bar{b}_{\tilde{\alpha}, \alpha}] & , \alpha > h(\underline{B}_{\tilde{\alpha}}) \end{cases} \right)$$

Then the arithmetic operations between these set can be performed as follows:

$$\tilde{A} \circ \tilde{B} = \bigcup_{\forall \tilde{\alpha}} \tilde{\alpha} \bigcup_{\forall \alpha} \alpha (\tilde{A}_{\tilde{\alpha}, \alpha} \circ \tilde{B}_{\tilde{\alpha}, \alpha}) \quad (5.13)$$

where the arithmetic between each T2FS  $\alpha$ -cut can be calculated as follows:

$$\tilde{A}_{\tilde{\alpha}, \alpha} \circ \tilde{B}_{\tilde{\alpha}, \alpha} = \left( \begin{cases} [\underline{a}_{\tilde{\alpha}, \alpha}, \underline{a}_{\tilde{\alpha}, \alpha}] \circ [\underline{b}_{\tilde{\alpha}, \alpha}, \underline{b}_{\tilde{\alpha}, \alpha}], [\bar{a}_{\tilde{\alpha}, \alpha}, \bar{a}_{\tilde{\alpha}, \alpha}] \circ [\bar{b}_{\tilde{\alpha}, \alpha}, \bar{b}_{\tilde{\alpha}, \alpha}] & , \alpha \leq h_{min} \\ \emptyset, [\bar{a}_{\tilde{\alpha}, \alpha}, \bar{a}_{\tilde{\alpha}, \alpha}] \circ [\bar{b}_{\tilde{\alpha}, \alpha}, \bar{b}_{\tilde{\alpha}, \alpha}] & , \alpha > h_{min} \end{cases} \right)$$

and  $h_{min}$  is the minimum of the heights of the LMFs of each  $\alpha$ -plane, i.e.,

$$h_{min} = \min(h(\underline{A}_{\tilde{\alpha}}), h(\underline{B}_{\tilde{\alpha}}))$$

This derivation subsumes all kinds of T2FNs i.e. perfect or semi-perfect as well. Finally, consider the special case of QT2FNs where only two  $\alpha$ -planes are involved in the computation, the FOU at  $\tilde{\alpha} = 0$  and the PS at  $\tilde{\alpha} = 1$ . Then we can use for perfectly normal T2FNs or equation for normal T2FNs. Let  $\tilde{A} = \langle \tilde{A}_0, \tilde{A}_1 \rangle$  and  $\tilde{B} = \langle \tilde{B}_0, \tilde{B}_1 \rangle$  be two QT2FNs completely determined by their FOU and PS. Then the result of basic arithmetic operations between them is another QT2FN shown as follows:

$$\tilde{A} \circ \tilde{B} = \langle \tilde{A}_0 \circ \tilde{B}_0, \tilde{A}_1 \circ \tilde{B}_1 \rangle \quad (5.14)$$

it has to be noted that to perform these operations the methods applied to IVFNs is used, i.e., Eq. 5.11.

**Example 5.2.1** In this example a QT2FN is considered, it also gives sufficient insight on more general T2FN arithmetic. Let the following triangular QT2FN  $\tilde{3} = \langle \tilde{3}_0, \tilde{3}_1 \rangle$  depicted in Figure 5.1 with parameters derived from Definition 5.1.10, i.e.,

$$\tilde{3}_0 = \langle 1.5, 2.25, 3, 3.45, 4.75, 0.6 \rangle, \text{ and } \tilde{3}_1 = \langle 1.75, 3, 4.25 \rangle$$

Notice that  $\tilde{3}$  is a semi-perfect T2FN with  $h(\tilde{3}_0) = 0.6$  and  $\tilde{3}_1$  is a perfect IVFN, in fact it is FN. Let also another triangular QT2FN  $\tilde{12} = \langle \tilde{12}_0, \tilde{12}_1 \rangle$  depicted in Figure 5.2 with parameters:

$$\tilde{12}_0 = \langle 10.25, 11.5, 12, 12.5, 14, 0.7 \rangle, \text{ and } \tilde{12}_1 = \langle 0.75, 12, 13.5 \rangle$$

Notice that  $\tilde{3}$  is a semi-perfect T2FN with  $h(\tilde{12}_0) = 0.7$  and  $\tilde{3}_1$  is also a FN. When computing the addition  $\tilde{3} + \tilde{12}$ , first, determine a suitable number of  $\alpha$ -cuts along  $U$  for both FOU and  $PS^*$ . For the sake of clarity  $U$  is discretised into 25  $\alpha$ -cuts. Then, applying Eq. 5.14 to these T2FNs gives the result QT2FN  $\tilde{15}$  and depicted in Figure 5.3 with the following parameters:

$$\tilde{15}_0 = \langle 11.75, 13.75, 15, 15.95, 18.75, 0.6 \rangle, \text{ and } \tilde{15}_1 = \langle 12.5, 15, 17.75 \rangle$$

The result is as expected from this addition,  $\tilde{3} + \tilde{12} = \tilde{15}$ .

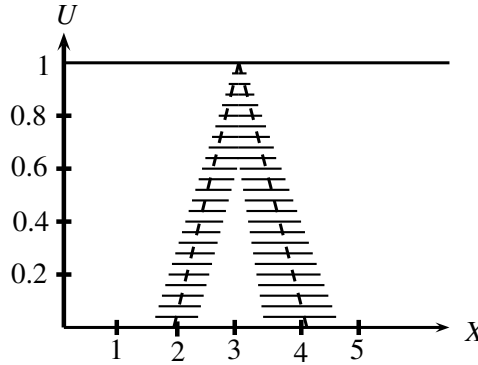


Fig. 5.1. QT2FN  $\tilde{3}$  with discretised  $\alpha$ -cuts of the  $FOU(\tilde{3})$ , and the dashed line is  $PS(\tilde{3})$

The following example shows how to perform addition of IVFNs using  $\alpha$ -cuts.

---

\*In the case of T2FN, first discretise along  $\tilde{U}$  in order to determine a suitable number of  $\alpha$ -planes, then discretise along  $U$  for each of the  $\alpha$ -planes.



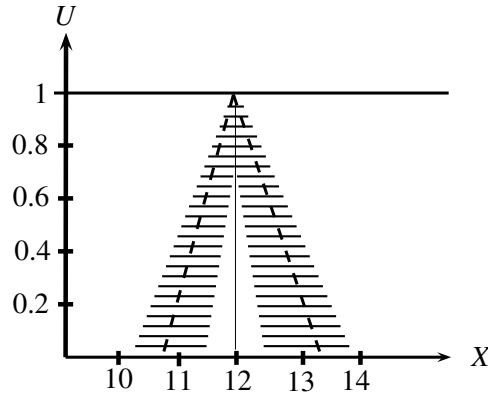


Fig. 5.2. QT2FN  $\widetilde{12}$  with discretised  $\alpha$ -cuts of the FOU  $FOU(\widetilde{12})$ , and the dashed line is  $PS(\widetilde{12})$

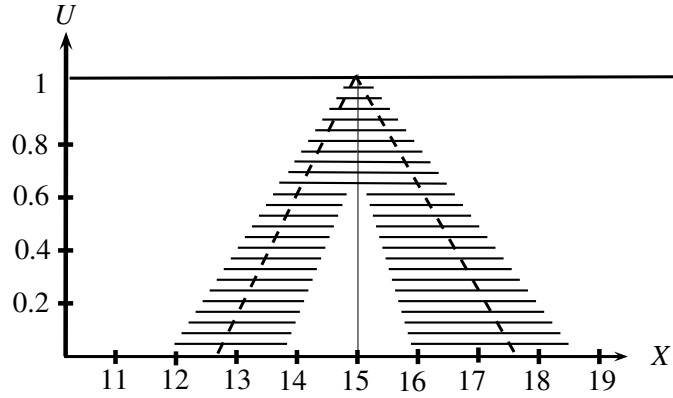


Fig. 5.3. QT2FN  $\widetilde{15}$  with discretised  $\alpha$ -cuts of the FOU  $FOU(\widetilde{15})$ , and the dashed line is  $PS(\widetilde{15})$ .

**Example 5.2.2** Let  $\widehat{4}$  and  $\widehat{8}$  be two IVFS defined in Table 5.1 and Table 5.2, respectively. The  $\alpha$ -cuts of both their LMF and UMF is shown in Table 5.3. The addition of the  $\alpha$ -cuts are shown in Table 5.4. This will eventually lead to an IVFS  $\widehat{12}$ . The method used to generate the membership grades of  $\widehat{12}$  from its  $\alpha$ -cuts is shown in Table 5.5.

Table 5.1. IVFS, $\widehat{4}$ , in example (5.2.2).					
$x$	2	3	4	5	6
$\widehat{4}(x)$	[0, 0.2]	[0.4, 0.6]	[0.8, 1]	[0.5, 0.6]	[0, 0.4]

Table 5.2. IVFS,  $\widehat{8}$ , in example (5.2.2).

$x$	5	6	7	8	9	10	11
$\widehat{8}(x)$	[0, 0.1]	[0.2, 0.5]	[0.6, 0.8]	[1, 1]	[0.5, 0.8]	[0.2, 0.4]	[0, 0.1]

Table 5.3. The  $\alpha$ -cuts of IVFS,  $\widehat{4}$  and  $\widehat{8}$ , in example (5.2.2).

$\alpha$	$\widehat{4}_\alpha$	$\widehat{8}_\alpha$	$\widehat{4}_\alpha$	$\widehat{8}_\alpha$
0.0	{2, 3, 4, 5, 6}	{5, 6, 7, 8, 9, 10, 11}	{2, 3, 4, 5, 6}	{5, 6, 7, 8, 9, 10, 11}
0.1	{3, 4, 5}	{6, 7, 8, 9, 10}	{2, 3, 4, 5, 6}	{5, 6, 7, 8, 9, 10, 11}
0.2	{3, 4, 5}	{6, 7, 8, 9, 10}	{2, 3, 4, 5, 6}	{6, 7, 8, 9, 10}
0.3	{3, 4, 5}	{7, 8, 9}	{3, 4, 5, 6}	{6, 7, 8, 9, 10}
0.4	{3, 4, 5}	{7, 8, 9}	{3, 4, 5, 6}	{6, 7, 8, 9, 10}
0.5	{4, 5}	{7, 8, 9}	{3, 4, 5}	{6, 7, 8, 9}
0.6	{4}	{7, 8}	{3, 4, 5}	{7, 8, 9}
0.7	{4}	{8}	{4}	{7, 8, 9}
0.8	{4}	{8}	{4}	{7, 8, 9}
0.9	$\emptyset$	{8}	{4}	{8}
1.0	$\emptyset$	{8}	{4}	{8}

We show here how convenient it is and how to use it by examining the following arithmetic example.

**Example 5.2.3** *Coupland & John (2003) and Blewitt et al. (2007) used an example of T2FS addition  $\widetilde{3} + \widetilde{12}$ . We interpret the same sets in Table 5.6 and Table 5.7. To perform this addition we need to decompose each T2FS to its  $\alpha$ -planes and each  $\alpha$ -plane to its  $\alpha$ -cuts. We take for example the addition of  $\alpha$ -planes  $\widetilde{3}_{0.2} + \widetilde{12}_{0.2}$ , we construct the interval membership grade of each  $\alpha$ -plane using the bounds of the PMs  $J_{x,\widetilde{\alpha}}$ , i.e. Table 5.8 and Table 5.9. The steps to perform the addition is shown in Table 5.10, Table 5.11 and Table 5.12. These are the same steps used to add IVFSs. To perform the addition of the T2FSs we have to perform the same task to all  $\alpha$ -planes. The final result is shown in Table 5.13.*

Table 5.4. The  $\alpha$ -cuts of IVFS,  $\widehat{12} = \widehat{4} + \widehat{8}$ , in example (5.2.2).

$\alpha$	$\underline{12}_\alpha = \underline{4}_\alpha + \underline{8}_\alpha$	$\overline{12}_\alpha = \overline{4}_\alpha + \overline{8}_\alpha$
0.0	{7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17}	{7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17}
0.1	{9, 10, 11, 12, 13, 14, 15}	{7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17}
0.2	{9, 10, 11, 12, 13, 14, 15}	{8, 9, 10, 11, 12, 13, 14, 15, 16}
0.3	{10, 11, 12, 13, 14}	{9, 10, 11, 12, 13, 14, 15, 16}
0.4	{10, 11, 12, 13, 14}	{9, 10, 11, 12, 13, 14, 15, 16}
0.5	{11, 12, 13, 14}	{9, 10, 11, 12, 13, 14}
0.6	{11, 12}	{10, 11, 12, 13, 14}
0.7	{12}	{11, 12, 13}
0.8	{12}	{11, 12, 13}
0.9	$\emptyset$	{12}
1.0	$\emptyset$	{12}

Table 5.5. IVFS,  $\widehat{12}$ , in example (5.2.2) from its  $\alpha$ -cuts in table (5.4)

$x$	$\alpha \underline{12}_\alpha(x)$	$\alpha \overline{12}_\alpha(x)$	$\widehat{12}(x)$
7	0	0, 0.1	[0, 0.1]
8	0	0, 0.1, 0.2	[0, 0.2]
9	0, 0.1, 0.2	0, 0.1, 0.2, 0.3, 0.4, 0.5	[0.2, 0.5]
10	0, 0.1, 0.2, 0.3, 0.4	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6	[0.4, 0.6]
11	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8	[0.6, 0.8]
12	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1	[0.8, 1]
13	0, 0.1, 0.2, 0.3, 0.4, 0.5	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8	[0.5, 0.8]
14	0, 0.1, 0.2, 0.3, 0.4, 0.5	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6	[0.5, 0.6]
15	0, 0.1, 0.2	0, 0.1, 0.2, 0.3, 0.4	[0.2, 0.4]
16	0	0, 0.1, 0.2, 0.3, 0.4	[0, 0.4]
17	0	0, 0.1	[0, 0.1]

### 5.3 Summary

In this chapter an investigation on type-2 fuzzy numbers and some operations that include such numbers are discussed including basic arithmetic operations. The basic arithmetic operations presented in this chapter include addition, subtraction, multiplication and division. These basic operations pave the way for many other operation, in fact the complete field of mathematics of classical numbers can be extended to even higher levels of uncertainty such as interval valued fuzzy sets and type-2 fuzzy sets. These numbers are presented In this chapter only the  $\alpha$ -cut representation theorem and the  $\alpha$ -cut extension principle are used in defining these novel operations. The applications of the arithmetic operations presented in this chapter are already well documented in the

Table 5.6. T2FS  $\widetilde{3}$ , in Example 5.2.3. The numbers in between are the SGs,  $\widetilde{3}_x(u_x)$ .

$x/u_x$	<b>0.0</b>	<b>0.1</b>	<b>0.2</b>	<b>0.3</b>	<b>0.4</b>	<b>0.5</b>	<b>0.6</b>	<b>0.7</b>	<b>0.8</b>	<b>0.9</b>	<b>1.0</b>
<b>1</b>	1.0	0.6	0.3								
<b>2</b>				0.1	0.6	1.0	0.7	0.2			
<b>3</b>											1.0
<b>4</b>				0.1	0.6	1.0	0.7	0.2			
<b>5</b>	1.0	0.6	0.3								

Table 5.7. T2FS  $\widetilde{12}$ , in Example 5.2.3. The numbers in the body of the table are the SGs,  $\widetilde{12}_x(u_x)$ .

$x/u_x$	<b>0.0</b>	<b>0.1</b>	<b>0.2</b>	<b>0.3</b>	<b>0.4</b>	<b>0.5</b>	<b>0.6</b>	<b>0.7</b>	<b>0.8</b>	<b>0.9</b>	<b>1.0</b>
<b>10</b>	1.0	0.8	0.4	0.2	0.1						
<b>11</b>						0.2	1.0	0.4			
<b>12</b>											1.0
<b>13</b>						0.2	1.0	0.4			
<b>14</b>	1.0	0.8	0.4	0.2	0.1						

literature of fuzzy sets. To summarise, the following table shows the contributions presented in this chapter:

Table 5.8.  $\alpha$ -plane,  $\widehat{3}_{0.2}$ , in Example 5.2.3.

$x$	1	2	3	4	5
$\widehat{3}_{0.2}(x)$	[0, 0.2]	[0.4, 0.7]	[1, 1]	[0.4, 0.7]	[0, 0.2]

Table 5.9.  $\alpha$ -plane,  $\widetilde{12}_{0.2}$ , in Example 5.2.3.

$x$	10	11	12	13	14
$\widetilde{12}_{0.2}(x)$	[0, 0.3]	[0.5, 0.7]	[1, 1]	[0.5, 0.7]	[0, 0.3]

Table 5.10. The  $\alpha$ -cuts of  $\alpha$ -planes,  $\widehat{3}_{0.2}$  and  $\widetilde{12}_{0.2}$ , in Example 5.2.3.

$\alpha$	$\widehat{3}_{0.2,\alpha}$	$\widetilde{12}_{0.2,\alpha}$	$\widehat{3}_{0.2,\alpha}$	$\widetilde{12}_{0.2,\alpha}$
0.0	{1, 2, 3, 4, 5}	{10, 11, 12, 13, 14}	{1, 2, 3, 4, 5}	{10, 11, 12, 13, 14}
0.1	{2, 3, 4}	{11, 12, 13}	{1, 2, 3, 4, 5}	{10, 11, 12, 13, 14}
0.2	{2, 3, 4}	{11, 12, 13}	{1, 2, 3, 4, 5}	{10, 11, 12, 13, 14}
0.3	{2, 3, 4}	{11, 12, 13}	{2, 3, 4}	{10, 11, 12, 13, 14}
0.4	{2, 3, 4}	{11, 12, 13}	{2, 3, 4}	{11, 12, 13}
0.5	{3}	{11, 12, 13}	{2, 3, 4}	{11, 12, 13}
0.6	{3}	{12}	{2, 3, 4}	{11, 12, 13}
0.7	{3}	{12}	{2, 3, 4}	{11, 12, 13}
0.8	{3}	{12}	{3}	{12}
0.9	{3}	{12}	{3}	{12}
1.0	{3}	{12}	{3}	{12}

Table 5.11. The  $\alpha$ -cuts of  $\alpha$ -planes,  $\widetilde{15}_{0.2} = \widehat{3}_{0.2} + \widetilde{12}_{0.2}$ , in Example 5.2.3.

$\alpha$	$\widetilde{15}_{0.2,\alpha} = \widehat{3}_{0.2,\alpha} + \widetilde{12}_{0.2,\alpha}$	$\widetilde{15}_{0.2,\alpha} = \widehat{3}_{0.2,\alpha} + \widetilde{12}_{0.2,\alpha}$
0.0	{11, 12, 13, 14, 15, 16, 17, 18, 19}	{11, 12, 13, 14, 15, 16, 17, 18, 19}
0.1	{13, 14, 15, 16, 17}	{11, 12, 13, 14, 15, 16, 17, 18, 19}
0.2	{13, 14, 15, 16, 17}	{11, 12, 13, 14, 15, 16, 17, 18, 19}
0.3	{13, 14, 15, 16, 17}	{12, 13, 14, 15, 16, 17, 18}
0.4	{13, 14, 15, 16, 17}	{13, 14, 15, 16, 17}
0.5	{14, 15, 16}	{13, 14, 15, 16, 17}
0.6	{15}	{13, 14, 15, 16, 17}
0.7	{15}	{13, 14, 15, 16, 17}
0.8	{15}	{15}
0.9	{15}	{15}
1.0	{15}	{15}

Table 5.12.  $\alpha$ -plane,  $\widetilde{15}_{0.2}$ , in Example 5.2.3 from its  $\alpha$ -cuts in Table 5.11.

$x$	$\alpha \widetilde{15}_{0.2, \alpha}(x)$	$\alpha \overline{\widetilde{15}}_{0.2, \alpha}(x)$	$\widetilde{15}_{0.2}(x)$
11	0	0, 0.1, 0.2	[0, 0.2]
12	0	0, 0.1, 0.2, 0.3	[0, 0.3]
13	0, 0.1, 0.2, 0.3, 0.4	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7	[0.4, 0.7]
14	0, 0.1, 0.2, 0.3, 0.4, 0.5	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7	[0.5, 0.7]
15	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1	[1, 1]
16	0, 0.1, 0.2, 0.3, 0.4, 0.5	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7	[0.5, 0.7]
17	0, 0.1, 0.2, 0.3, 0.4	0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7	[0.4, 0.7]
18	0	0, 0.1, 0.2, 0.3	[0, 0.3]
19	0	0, 0.1, 0.2	[0, 0.2]

Table 5.13. The final result of  $\widetilde{3} + \widetilde{12} = \widetilde{15}$  in Example 5.2.3.

$x/u_x$	<b>0.0</b>	<b>0.1</b>	<b>0.2</b>	<b>0.3</b>	<b>0.4</b>	<b>0.5</b>	<b>0.6</b>	<b>0.7</b>	<b>0.8</b>	<b>0.9</b>	<b>1.0</b>
<b>11</b>	1.0	0.6	0.3								
<b>12</b>	1.0	0.8	0.4	0.2	0.1						
<b>13</b>				0.1	0.6	1.0	0.7	0.2			
<b>14</b>						0.2	1.0	0.4			
<b>15</b>											1.0
<b>16</b>						0.2	1.0	0.4			
<b>17</b>				0.1	0.6	1.0	0.7	0.2			
<b>18</b>	1.0	0.8	0.4	0.2	0.1						
<b>19</b>	1.0	0.6	0.3								

Table 5.14. The contributions presented in this chapter.

Concept	Contribution	Reference
Interval valued fuzzy numbers	New alternative	Definition 5.1.5
Type-2 fuzzy numbers	Novel	Definition 5.1.7 , 5.1.8 and 5.1.12
Arithmetic of IVFNs using $\alpha$ -cuts	New alternative	Eq. 5.12
Arithmetic of T2FNs using $\alpha$ -cuts	Novel	Eq. 5.13

## Chapter 6

# Parallel Computation and T2FSs

In Chapter 3, the  $\alpha$ -cut decomposition theorem was presented. This representation decomposes T2FSs into a collection of independent classical sets. Each of these crisp sets are associated with the original T2FS by two levels,  $\tilde{\alpha}$  and  $\alpha$ . These levels can be seen as an address of the  $\alpha$ -cut in the original set. It has also been shown in Chapter 3 that functions and operations on T2FSs can be achieved by applying the crisp version of these operations on all the  $\alpha$ -cuts of the T2FS. Chapter 4 showed this fact by defining measures of uncertainty for T2FSs, and Chapter 5 defined arithmetic operation in the same manner. This chapter explores the capability of processing operations using the  $\alpha$ -cut EP in parallel.

It is argued in this chapter that since these  $\alpha$ -cuts are crisp sets, and at the same time independent of each other, they have the potential of being successfully applied to massively parallel processing units such as Graphical Processing Units (GPUs). Computing functions and operations on T2FSs is a computationally expensive task due to the complex 3D nature of the T2FS. This high computational cost have been blamed for hindering the progress of the field of T2FL. The extra degrees of freedom offered by the T2FSs did not compensate for the high cost of computation to justify the implementation of T2FSs in practical applications. Recently the development of new high speed technologies in the microprocessors industry renewed the interest on the extra dimension provided by T2FSs. New algorithms are developed to aid in processing T2FS operations, and still no dramatic improvements have been reported that allow T2FSs be applicable on a large scale. One of the main advancements in technology is the utilisation of parallel processing both in multi-core CPUs and even more the massively parallel GPU capabilities. To be able to make use of this emergent and growing field of application, the input structure to these parallel processing units should be independent parallel structures (NVIDIA 2010). Recently GPU manufacturer giant NVIDIA released a new family of general purpose GPUs (GPGPUs), which targets applications other than graphics rendering. It is a highly parallel architecture with large number of multithreaded processors, which can significantly accelerate the performance of data-parallel applications (Wen-mei 2011). It is gaining popularity amongst various disciplines, ranging from

biology, physics, cosmology, medicine and etc. (see (Kirk & Wen-meì 2010) for a survey.) The boost in applications that involve GPU computing is due to the parallel programming model put forward by NVIDIA to aid the development of applications using NVIDIA's GPUs. The Compute Unified Device Architecture (CUDA) integration with C language provided an easy to use development tool that encourage researchers and developers to make use of the powerful computing power of the GPUs.

## 6.1 The Parallel Nature of the T2FS Alpha-cut RT

This section investigates mathematically the parallel nature of the  $\alpha$ -cut RT of T2FSs. The idea is to show that operations using  $\alpha$ -cuts are typically suitable for parallel processing. This suitability can be shown by examining the characteristics of typical parallel processing solutions. First, the optimum situation for parallel processing data on a GPU is having a Single Instruction Multiple Data (SIMD) structure (NVIDIA 2010). This means that the data units are independent of each other, and the single instruction is applied to all the data units as shown in Figure 6.1. This architecture is an exact description of the  $\alpha$ -cut extension principle for type-2 fuzzy sets defined in Chapter 3. Once each  $\alpha$ -cut is defined then operations on these  $\alpha$ -cuts are calculated independently. For example the addition of two T2FSs using the  $\alpha$ -cut RT require calculating the addition of the  $\alpha$ -cuts of the two T2FSs of the same level. This fact can be seen in Figure 6.2, which shows that it is a direct implementation of the SIMD architecture of Figure 6.1. On the other hand, examining the same capability for the  $\alpha$ -planes yields an introduction of an extra level. This is shown in Figure 6.3, this extra level is required to perform the iterative procedure for the addition of the IVFS  $\alpha$ -planes using the point-wise extension principle. This extension principle requires for each point computing the supremum of the points and the minimum of the corresponding secondary grades. This extra level of operations are not needed for the  $\alpha$ -cut RT of T2FSs. The introduction of an iteration procedure makes computation very difficult for the GPUs. If this is the case with the some what parallel  $\alpha$ -plane RT, it is even more difficult to apply the point-wise EP for T2FSs, in fact it is a waste of time to compute all the iterations involved for this operation. Any function or operation is either a single instruction or can be divided in to several single instructions applied to or between the independent  $\alpha$ -cuts. The  $\alpha$ -plane representation theorem and the  $\alpha$ -plane extension principle defined in Chapter 3 also have a level of independence between the  $\alpha$ -planes. The structure of the  $\alpha$ -planes themselves is complicated, they are IVFSs. The operations on the  $\alpha$ -planes themselves, in most cases, require the use of the extension principle for IVFSs. To explain this issue, assume that two T2FSs  $\tilde{A}$  and  $\tilde{B}$  are both decomposed into  $M$   $\alpha$ -planes, and  $M \times N$   $\alpha$ -cuts. Let also  $f$  be a binary crisp function that takes two values and produces one value.



Using the  $\alpha$ -plane extension:

$$f(\tilde{A}, \tilde{B}) = \bigcup_{i=1 \dots M} \tilde{\alpha}_i(f(\tilde{A}_{\tilde{\alpha}_i}, \tilde{B}_{\tilde{\alpha}_i}))$$

then the extension of  $f$  to a function between IVFSs is performed  $M$  times, and the extension of  $f$  to IVFSs is calculated using the classical extension principle or by using the embedded standard fuzzy set extension. On the other hand, using the  $\alpha$ -cut extension principle:

$$f(\tilde{A}, \tilde{B}) = \bigcup_{i=1 \dots M} \tilde{\alpha}_i \bigcup_{j=1 \dots N} \alpha_j(f(\tilde{A}_{\tilde{\alpha}_i, \alpha_j}, \tilde{B}_{\tilde{\alpha}_i, \alpha_j}))$$

then  $f$  is performed  $M \times N$  times on the pair of classical sets, which turn out to be  $2 \times M \times N$  times. It is clear that the  $\alpha$ -cut extension principle is highly parallel in nature, and given the volume of data to be processed using a single instruction makes it suitable for GPU computing. The following section gives some basic insight on the GPU computing architecture.

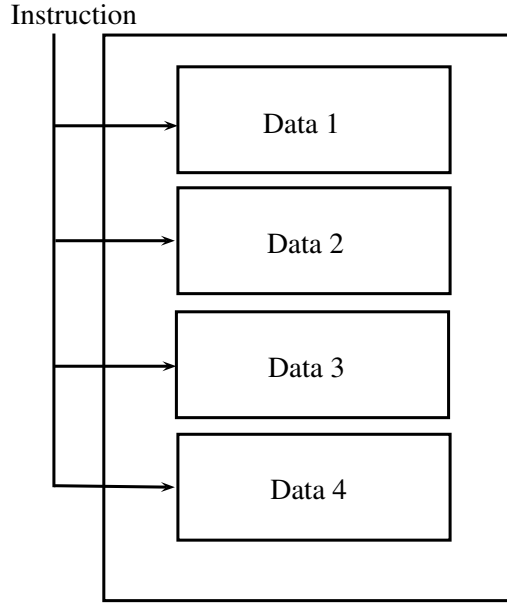


Fig. 6.1. SIMD architecture.

## 6.2 Experimental Design

The experiment presented in this chapter is designed to evaluate the performance of a GPU implementation of T2FS operations compared to a CPU implementation. The CPU, GPU and soft-

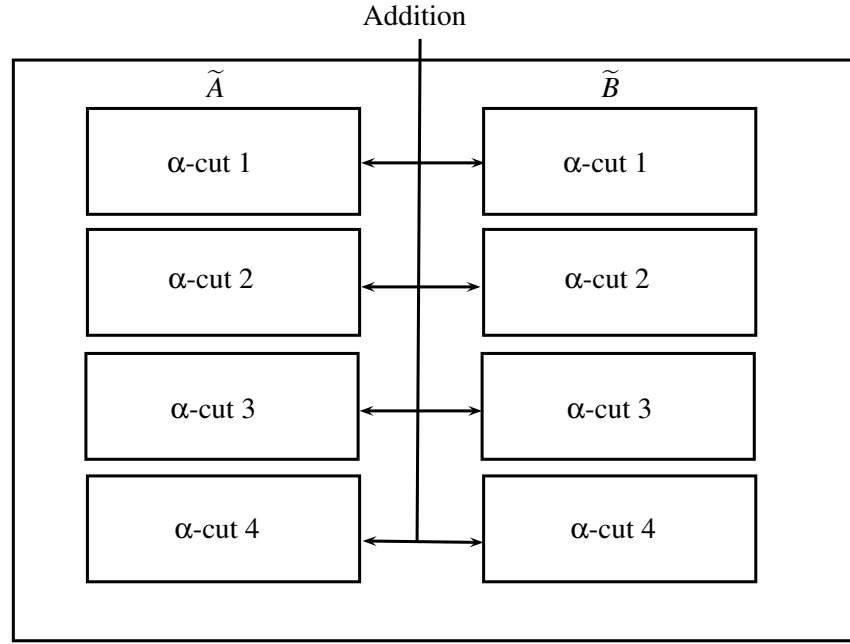


Fig. 6.2. The addition of two T2FSs using the  $\alpha$ -cut RT.

ware/hardware specifications of the test environment is summarised in Table 6.1. Appendix A is a primer on GPU architecture and CUDA parallel programming model. The details regarding the technical specifications and software considerations have been explained in the Appendix and in this chapter only the experiments, result and analysis are discussed. Four different comparisons are performed with different aims.

- **Experiment 1:** The performance of the GPU is compared against the CPU using the  $\alpha$ -cut representation theorem. The addition of two type-2 fuzzy sets are chosen for this test because it is simple and straight forward. The idea behind this test is to evaluate the performance of the  $\alpha$ -cut representation theorem and exploit its parallel nature with a very simple function such as addition which does not require high processing power.
- **Experiment 2:** The performance of GPU processing is compared against the CPU using the  $\alpha$ -cut representation theorem. The addition of two type-2 fuzzy sets is used again. The difference between this test and the first test is the measure of performance of the GPU. The time calculated in this test is the time of processing only. The time of data transfer is excluded from measurement. The aim of this test is to evaluate the effect of data transfer on the GPU performance.
- **Experiment 3:** The same test is performed as the first test, but this time the simple addition

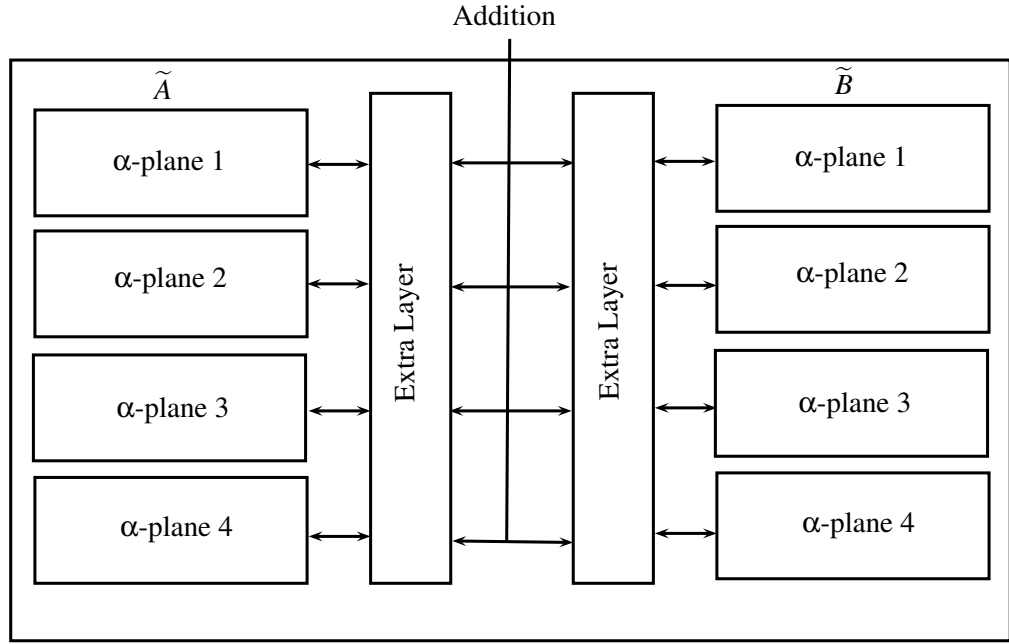


Fig. 6.3. The addition of two T2FSs using the  $\alpha$ -plane RT.

operation is replaced with more than one arithmetic operation. The following arithmetic operation is calculated for three type-2 fuzzy set,  $(\tilde{A} + \tilde{B}) \times \tilde{C}$ , using the the  $\alpha$ -cut representation theorem. The  $\alpha$ -cut representation theorem is also chosen, and are processed through the GPU and the CPU. The aim of this test is to visualise the effect of raising the complexity of operations on the performance gain between the GPU and CPU.

- **Experiment 4:** The following operation is calculated,  $(\tilde{A} \cap \tilde{B}) \cup \tilde{C}$ , it involves the union and intersection of three type-2 fuzzy sets. This time the  $\alpha$ -cut representation theorem is compared against the  $\alpha$ -plane representation theorem. The  $\alpha$ -cut version is processed on the GPU and the  $\alpha$ -plane version is processed on the CPU. The aim of this test is to evaluate the performance of  $\alpha$ -cut implementations against  $\alpha$ -plane implementations.

In all the above, the evaluation is kept simple and clear. The time elapsed difference is calculated, and mean of 30 runs of each calculation is recorded and compared. The parameters of the type-2 fuzzy sets are generated randomly with sensible constraints. These type-2 fuzzy sets are Quasi T2FNs\*, and the generated parameters are described by Eq. 6.1 and shown in Figure 6.4.

$$\langle FOU, PS \rangle = \langle \underline{s}_0, \underline{m}_0, \underline{e}_0, \underline{h}_0, \bar{s}_0, \bar{m}_0, \bar{e}_0, \underline{s}_1, \underline{m}_1, \underline{e}_1, \underline{h}_1, \bar{s}_1, \bar{m}_1, \bar{e}_1 \rangle \quad (6.1)$$

\*see Definition 5.1.12 for details.

The discretisation rate across the primary axis,  $U$ , and secondary axis  $\tilde{U}$  is calculated according to Table 6.2. The number of  $\alpha$ -cuts equals ( $2 \times$  the number of discrete points in  $U \times$  the number of discrete points in  $\tilde{U}$ ) because the  $\alpha$ -cuts are calculated from the LMF and UMF. On the other hand the number of  $\alpha$ -planes is equal to the number of discrete points in  $\tilde{U}$ . The evaluation is calculated by the following formula:

$$E = \frac{\text{CPU Time} - \text{GPU Time}}{\text{CPU Time}} \times 100 \quad (6.2)$$

where  $E$  is the percentage of gain in performance made by the GPU over the CPU, obviously if  $E$  is negative it shows a gain for the CPU over the GPU. A simple and clear criteria is followed in order to carry out these tests, and is summarised as follows:

1. Type-2 fuzzy set parameters are generated at random. The parameters are described as  $\langle FOU, PS \rangle$  which is defined in Eq. 6.1 and shown in Figure 6.4. It represents the FOU and the Principal Set of the T2FS.
2. Then each T2FS is discretised for each test according to Table 6.2. This method of discretisation is similar to the grid method presented by Greenfield et al. (2009).
3. The  $\alpha$ -cuts and  $\alpha$ -planes are determined and stored into appropriate vectors.
4. Depending on the operation, the vectors are passed for processing to the GPU and CPU. At this stage the time elapsed are measured and recorded.
5. The above are repeated 30 times, and the mean of the 30 measured processing times is calculated.

Note that the  $\alpha$ -plane implementation uses a discretisation across the domain of the type-2 fuzzy set. The discretisation level used for the domain valued is equal to the number of  $\alpha$ -cut discretisation. Next, the evaluation of the results are presented and discussed.

### 6.3 Results

The main purpose behind carrying out the aforementioned tests is to highlight the potential of practical application of T2FS. This is shown by taking advantage of the parallel nature of the novel  $\alpha$ -cut representation of T2FSs presented earlier in Chapter 3 of this thesis. The discussion in this section follows from the sequence presented in the previous section.

Table 6.1. Hardware and software specifications of the experiments.

OS	Windows Vista Home Premium, service pack 2
NVIDIA CUDA Technology	Toolkit 4.0
CUDA capability	1.1
Development environment	Microsoft Visual C++ 2008 SP1
Software libraries	Thrust v 1.4.0
CPU	Intel Core 2 Quad CPU
CPU Speed	2.66 GHz
CPU No of cores	4
RAM	4 GB
System type	32-bit
GPU	NVIDIA GeForce 9800 GX2
Graphics Clock	600 MHz
Processor Clock	1500 MHz
Texture Fill Rate	76.8 billion/sec
Memory Clock	1000 MHz
Max No of Threads per Block	512
Warp size	32
No of Multiprocessors	16
No of GPU devices	2
No of CUDA Cores	256 (128 per GPU device)

Table 6.2. Discretisation across the primary axis  $U$  and secondary axis  $\tilde{U}$ .

$U$	$\tilde{U}$	No of $\alpha$ -cuts	No of $\alpha$ -planes
8	8	128	8
16	8	256	8
16	16	512	16
32	16	1024	16
32	32	2048	32
64	32	4096	32

- **Experiment 1** calculates the addition of two type-2 fuzzy sets. Addition is a simple and straight forward operation, and the performance is measured by evaluating the addition of two vectors. The two vectors vary in length according to the level of discretisation. For example the 128  $\alpha$ -cuts involves  $128 \times 2 = 256$  addition operations. This is because each  $\alpha$ -cut is represented only using two numbers as the T2FS is assumed to be continuous for simplicity. The GPU is compared against the CPU using only the  $\alpha$ -cut representation theorem. The idea behind this test is to evaluate the performance of the  $\alpha$ -cut representation theorem and exploit its parallel nature with a very simple function such as addition which does not require high processing power. The results are shown in Figure 6.5 and Table 6.3. Figure 6.5 clearly shows significant increase in the performance gain of the GPU as the number of  $\alpha$ -cuts increase. This is an expected behaviour of GPU/CPU comparisons. The CPU outperforms the GPU when the number of operations are considerably low. The GPU begin to outperform the CPU as the number of operations increase. It is a matter of fact that in a continuous domain as the number of discretisation increase the finer and more accu-

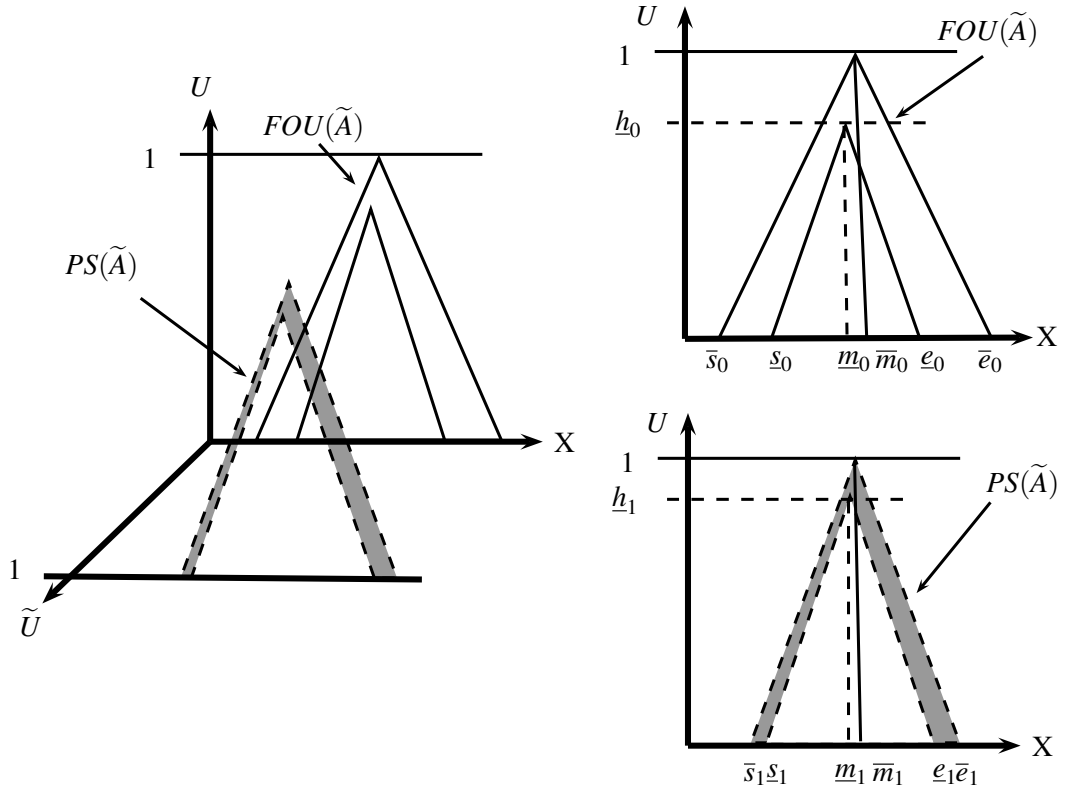


Fig. 6.4. The parameters of the T2FSs used in experiments.

rate the result is. The trade-off between accuracy and computational complexity is always a decisive issue, in real applications one must make a decision on whether accuracy or efficiency (computational) is of more importance. What the GPU in this test offers reasonably efficient alternative preserving reasonable accuracy represented by the greater number of discretisation points.

- **Experiment 2** calculates the addition of two type-2 fuzzy sets using the  $\alpha$ -cut representation theorem. The only difference between this test and the first test is the performance measure part. In this test the time the GPU takes in the addition operation is calculated without the time it takes to transfer the data from the CPU memory to the GPU memory. It is documented in the literature of GPU computing, e.g. (Nickolls & Dally 2010, Sanders & Kandrot 2010, Wen-mei 2011), that the bottle neck in GPU performance is caused by the delay in data transfer from CPU memory to GPU memory. The aim of this test is to evaluate the effect of this issue in the performance gain of the GPU over the CPU. The results of this test are presented in Figure 6.6 and Table 6.4. Figure 6.6 shows great GPU performance gain over that of the CPU. This is an expected behaviour since the addition is only performed

once on every thread. Consulting Table 6.4 also assures the same trend of proportional increase in performance gain with respect to the increase in the number of  $\alpha$ -cuts. This trend is expected to be valid for all tests that compare GPUs to CPUs. The main objective of this test is that if the bottle neck of data transfer is reduced the gain of performance of the GPU over the CPU is significantly high even with small number of data i.e.  $\alpha$ -cuts. This is clear from the first row of Table 6.4 which reported 92 % gain in performance in favour of the GPU with only 128  $\alpha$ -cuts.

- **Experiment 3** calculates the following formula,  $(\tilde{A} + \tilde{B}) \times \tilde{C}$ , between three type-2 fuzzy sets using the  $\alpha$ -cut representation theorem. The objective is to measure the performance of the GPU and CPU with a more complex operation. This calculation involves the addition and multiplication of the three type-2 fuzzy sets. The addition between  $\tilde{A}$  and  $\tilde{B}$  is calculated and the result is multiplied to  $\tilde{C}$ . The implementation of this function on the GPU involves invoking the GPU two times, one to calculate the addition operation and the other for computing the multiplication operation. The result is shown in Figure 6.7 and Table 6.5. Figure 6.7 verifies the same trend of proportional increase in performance gain of the GPU over the CPU with respect to the increase in the number of  $\alpha$ -cuts. The interesting part of this test is comparing Table 6.5 with Table 6.3. There is an increase in the performance gain of the GPU over the CPU in all the rows. This shows that the increase in the number of operations conducted in parallel using the GPU gives it increased performance gain over the CPU.
- **Experiment 4** calculates the following,  $(\tilde{A} \cap \tilde{B}) \cup \tilde{C}$ , which involves also three type-2 fuzzy sets. The  $\alpha$ -cut representation theorem over the GPU and the  $\alpha$ -plane representation theorem over the CPU are used as the basis for comparison. The objective is to compare the performance of the  $\alpha$ -cut representation theorem against the  $\alpha$ -plane representation theorem. The result is shown in Figure 6.8 and Table 6.6. Figure 6.8 shows the huge difference in performance gain between the GPU over the CPU. This is because the operations that involved the  $\alpha$ -plane representation theorem involved the use of the union and intersection of IVFSs. It has to be mentioned that the number of discretisation of the domain of the  $\alpha$ -planes is chosen to be the same number of  $\alpha$ -cuts for each run. Table 6.6 clearly shows that the  $\alpha$ -cut representation theorem outperforms the  $\alpha$ -plane representation theorem significantly. The fact that implementing this  $\alpha$ -plane require revisiting the memory which slows down the GPU performance.

Table 6.3. Adding two T2FSs using alpha-cuts in GPU and CPU.

Alpha-cuts	CPU Time	GPU Time	GPU gain %
128	0.00483	0.009468	-97
256	0.009827	0.015484	-58
512	0.020123	0.01898	6
1024	0.039842	0.031107	22
2048	0.078865	0.056751	29
4096	0.1613	0.092105	43

Table 6.4. Adding two T2FSs using alpha-cuts in GPU and CPU (GPU Time, processing only).

Alpha-cuts	CPU Time	GPU Time	GPU gain %
128	0.008908	0.005433	40
256	0.009688	0.005148	47
512	0.020284	0.005291	74
1024	0.047659	0.005512	89
2048	0.077672	0.006614	92
4096	0.156965	0.008136	95

## 6.4 Discussion

This chapter presented an investigation on the significant improvement in performance the the  $\alpha$ -cut representation theorem presented in this thesis can provide. The reason behind the improved performance is the ability to process type-2 fuzzy sets using graphical processing units for the first time. Four experiments are carried out to explore this capability and can be summarised as follows:

1. **Experiment 1:** The GPU is compared against the CPU using the  $\alpha$ -cut representation theorem to add two type-2 fuzzy sets.
2. **Experiment 2:** The GPU processing time only is compared against the CPU using also the  $\alpha$ -cut representation theorem to add two type-2 fuzzy sets.
3. **Experiment 3:** The GPU is compared against the CPU using the  $\alpha$ -cut representation theorem to calculate an operation that involves addition and multiplication, and three type-2 fuzzy sets.
4. **Experiment 4:** The  $\alpha$ -cut representation theorem on GPU is compared against the  $\alpha$ -plane on CPU to calculate an operation that involves union and intersection, and three type-2 fuzzy sets.



Table 6.5. The operation  $(\tilde{A} + \tilde{B}) \times \tilde{C}$  using  $\alpha$ -cuts processed on GPU and CPU.

Alpha-cuts	CPU Time	GPU Time	GPU gain %
128	0.01127	0.018877	-68
256	0.017758	0.019235	-9
512	0.035078	0.028526	19
1024	0.067378	0.045793	33
2048	0.136509	0.080956	41
4096	0.270125	0.150206	45

Table 6.6. The operation,  $(\tilde{A} \cap \tilde{B}) \cup \tilde{C}$ , using  $\alpha$ -cuts on the GPU and  $\alpha$ -planes on the CPU.

Alpha-cuts	CPU Time	GPU Time	GPU gain %
128	0.258035	0.021578	92
256	0.466861	0.021678	96
512	0.934989	0.029733	97
1024	1.864581	0.048964	98
2048	3.737532	0.078377	98
4096	7.441693	0.145887	99

These four tests in general showed the potential of the  $\alpha$ -cut representation of type-2 fuzzy sets for practical implementations. Overall, the  $\alpha$ -cut representation can be easily implemented on GPUs, in fact, it is natural and straight forward for  $\alpha$ -cuts to be processed in parallel. The results investigated in the previous section can be summarised as follows:

1. The performance gain of the GPU will always increase as the number of  $\alpha$ -cuts increase. This is shown across all the tests, and there has always been a threshold where the GPU will outperform the CPU.
2. The number of operations that can be processed in parallel also affects the performance gain of the GPU. Increasing the number of parallel operations increases the performance gain of the GPU over the CPU. This can be seen by comparing Experiment 3 and Experiment 1, in which percentage of performance gain in Experiment 3 is more than Experiment 1.
3. The processing time of the GPU is massively quicker than that of the CPU, and the delay in processing is only caused by the data transfer. This is clearly shown by comparing the results of Experiment 2 to the results of Experiment 1. The importance of this information comes from the fact that the increase in bandwidth and memory speed is the most significant factor in the overall processing time, something of a bottle neck. The bandwidth of the NVIDIA GTX 590 is 328320 MB/sec compared to 128000 MB/sec of the NVIDIA 9800 used in this thesis. This will prove to be pivotal to the increase in performance gain.
4. The comparison between any other representation and the  $\alpha$ -cut representation in terms of the speed of processing using the GPU proved to be extremely significant. The  $\alpha$ -plane representation is chosen because it also has some degree of parallelisation, but not much as

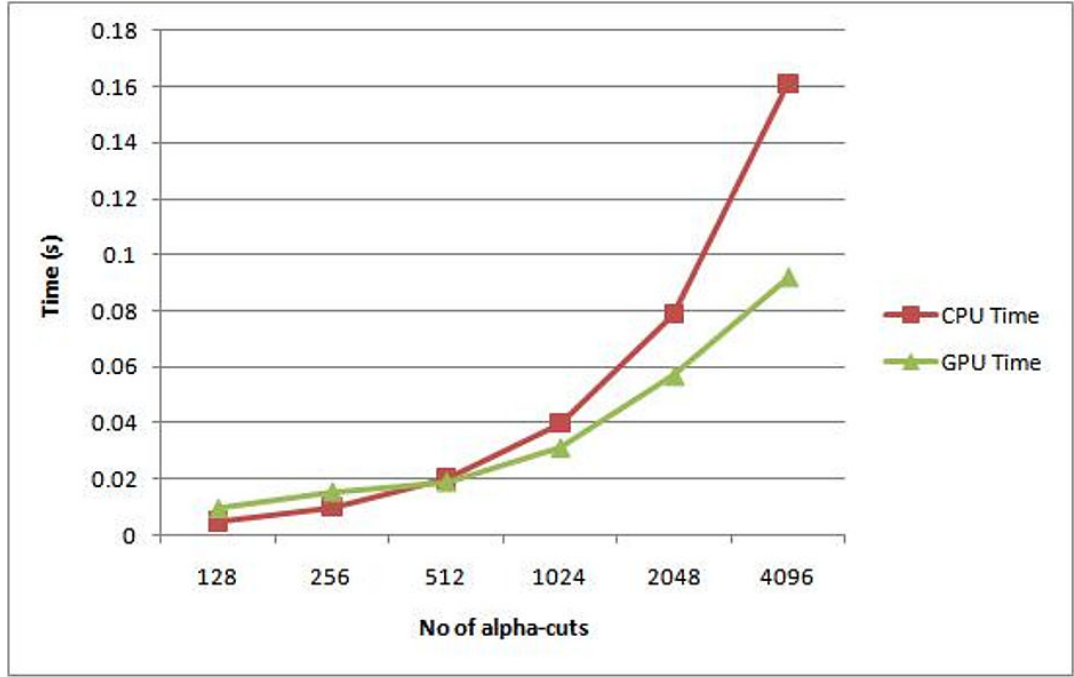


Fig. 6.5. Adding two T2FSs using alpha-cuts in GPU and CPU.

the  $\alpha$ -cut representation theorem. Each  $\alpha$ -plane may be processed independently, but still the algorithms used for interval valued fuzzy set calculations are not fully in parallel.

## 6.5 Summary

This chapter showed that the  $\alpha$ -cut representation theorem is extremely parallel. This fact makes the  $\alpha$ -cut extension principle extremely useful for practical applications, specially if a GPU can be used. To date no implementations of type-2 fuzzy sets on the GPU are found in the literature. To date the  $\alpha$ -cut representation of type-2 fuzzy sets is the only representation implemented on a GPU. In Chapter 3 a novel representation for type-2 fuzzy sets have been defined, along with a powerful extension principle that makes manipulating this representation very simple and straight forward. In Chapter 4 and Chapter 5, uncertainty measures and type-2 fuzzy numbers and arithmetic have been defined using the extension principles presented in this thesis. Finally, in this chapter the independent nature of the  $\alpha$ -cuts is proved to be useful for parallel processing in the massively

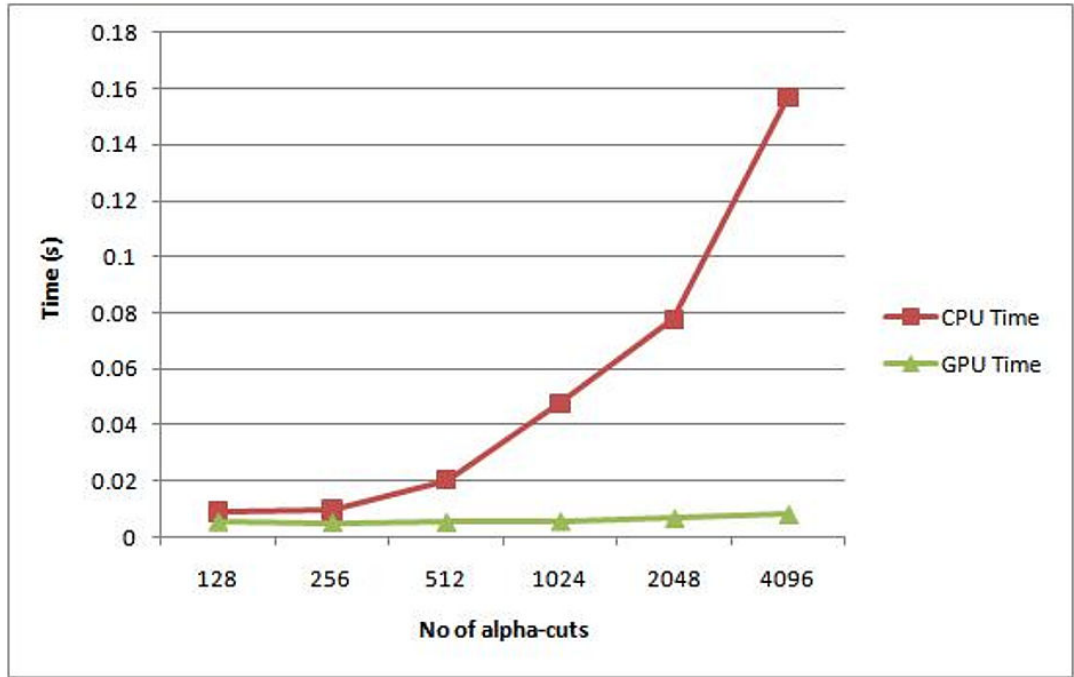


Fig. 6.6. Adding two T2FSs using alpha-cuts in GPU and CPU (GPU Time, processing only).

parallel GPUs, which shows great potential for practical applications. Overall, in Chapter 3, the  $\alpha$ -cut decomposition theorem was presented, and also shown that functions and operations on T2FSs can be extended from crisp sets directly. These definitions formed the basis for the theoretical and practical investigation presented in the rest of the thesis. In Chapter 4 measures of uncertainty for T2FSs was defined, and in Chapter 5 arithmetic operations were defined in the same manner. These two chapters demonstrated the theoretical advantage, i.e., the ease of defining operations for T2FSs using the  $\alpha$ -cut representation theorem. Finally, in this chapter massive computation performance provided by the  $\alpha$ -cut representation theorem was explained.

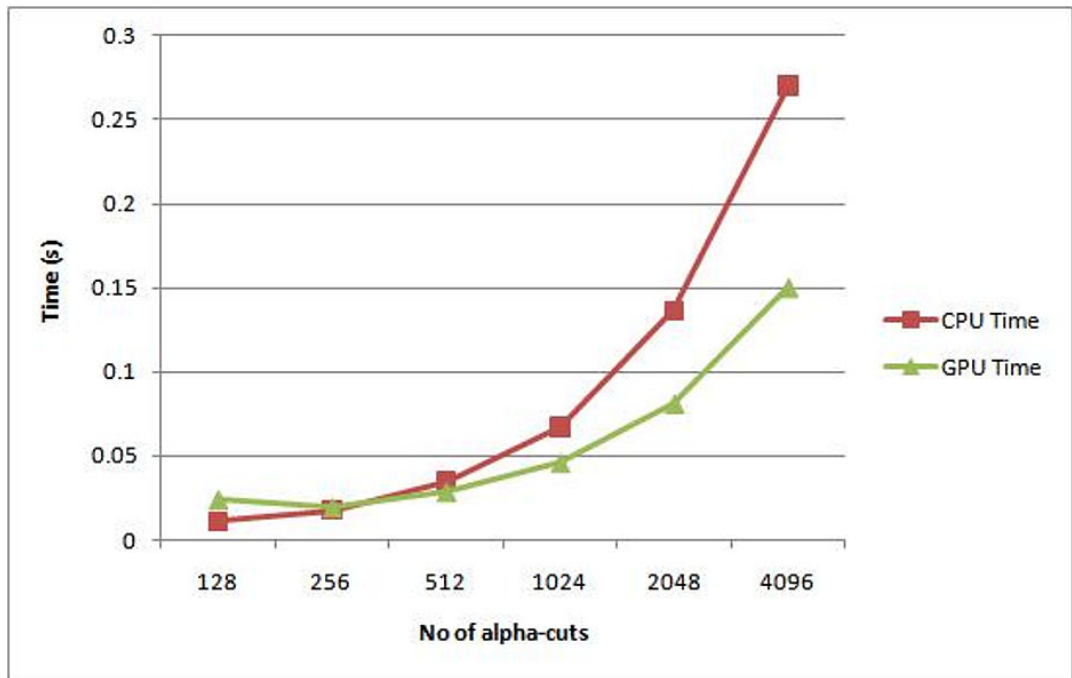


Fig. 6.7. The operation  $(\tilde{A} + \tilde{B}) \times \tilde{C}$  using  $\alpha$ -cuts processed on GPU and CPU.

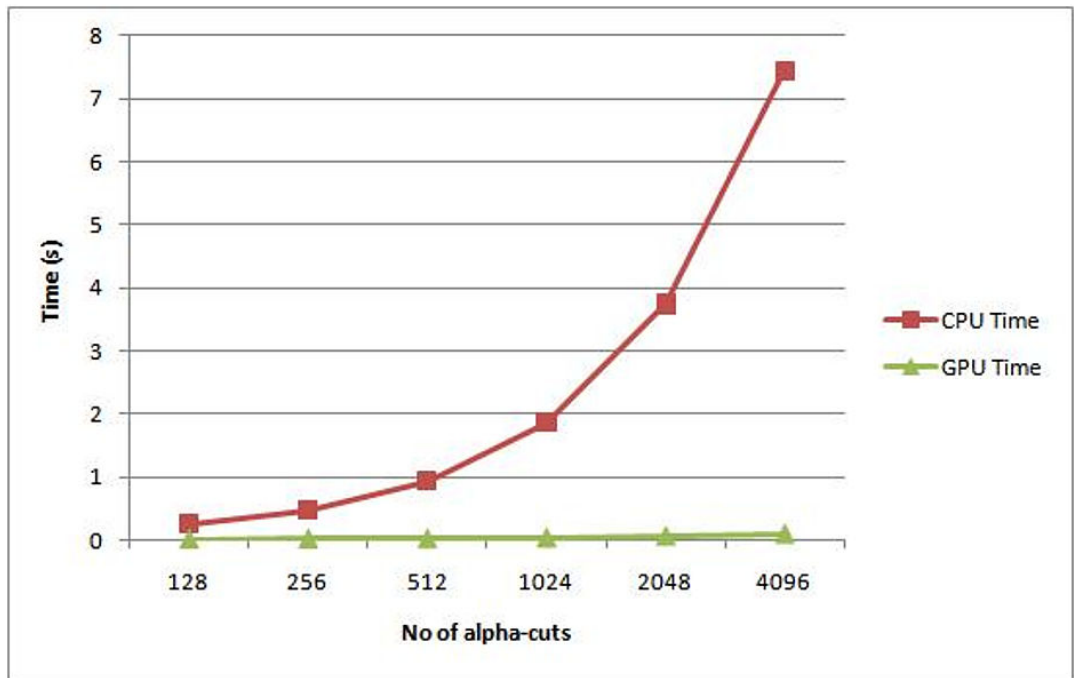


Fig. 6.8. The operation,  $(\tilde{A} \cap \tilde{B}) \cup \tilde{C}$ , using  $\alpha$ -cuts on the GPU and  $\alpha$ -planes on the CPU.

## Chapter 7

# Conclusion

This Chapter concludes the thesis by summarising the key points and outcomes of the research and discussing the directions for future work.

This thesis first started by discussing the notion of reasoning under uncertainty and its centrality for real world applications. It has been argued in Chapter 1 that uncertainty has many facets such as probability theory and fuzzy logic. The choice of fuzzy logic stems from the capability of fuzzy logic to model reality in human-like terms, which can be seen by the introduction of linguistic variables. On the other hand, it has been shown that extensions of fuzzy logic such as type-2 fuzzy logic model extra-levels of uncertainty. In general the role of type-2 fuzzy sets and its current state-of-the-art was discussed. Also Chapter 1 provided the objectives and motivation behind investigating type-2 fuzzy sets and its mathematical representation. It has been shown that type-2 fuzzy sets are not as popular as type-1 fuzzy sets in applications. The reason behind this was shown to be the complex mathematical nature of type-2 fuzzy sets compared to standard fuzzy sets. In order to overcome this obstacle, it has been argued that a suitable mathematical representation that is simple to manipulate with low computational cost is required.

This assertion was the starting point for Chapter 2 by the overview of the literature of standard fuzzy sets and its extensions. Concentration have been made towards type-2 fuzzy set representations in order to thoroughly study its mathematical structure. throughout this investigation careful emphasis has been put on the mathematical notations and definitions with an attempt to identify different relations and connections between different extension to fuzzy sets. The relation between classical sets, fuzzy sets, interval valued fuzzy sets and type-2 fuzzy sets has been highlighted.

Chapter 3 constituted the main outcomes of this thesis, namely, the  $\alpha$ -cut representation theorem of type-2 fuzzy sets. This representation theorem allowed the mathematical representation of type-2 fuzzy sets using a collection of classical sets. This significant step provided type-2 fuzzy logic with a powerful tool for mathematical manipulation. This  $\alpha$ -cut representation theorem made possible the direct extension of functions and operations from classical sets to type-2 fuzzy sets through the  $\alpha$ -cut extension principle. This principle made possible the definition of several

mathematical concepts for type-2 fuzzy sets.

In Chapter 4 different uncertainty measures have been defined utilising the  $\alpha$ -plane extension principle presented in Chapter 3. Cardinality, similarity, subethood, fuzziness and non-specificity measures have been defined for type-2 fuzzy sets. Mathematical proofs and axioms has been identified for type-2 fuzzy sets and interval valued fuzzy sets. This methodology has been shown to have the possibility to generalise to other functions and axioms.

The  $\alpha$ -cut representation theorem has been found very useful to define different kinds of type-2 fuzzy numbers. It also has been used to extend arithmetic operations from classical numbers to type-2 fuzzy numbers in a straight forward manner. The investigation in Chapter 5 has included some worked examples in order to show the usefulness of these arithmetic extensions. Also the new concept of Quasi type-2 fuzzy sets and numbers has been investigated.

Finally, in Chapter 6 the independent nature of each  $\alpha$ -cut of type-2 fuzzy sets allowed parallel processing be applied for type-2 fuzzy set operations for the first time. Various experiments has been used to demonstrate the ability and impact of the  $\alpha$ -cut representation theorem on type-2 fuzzy sets. Computational cost has been measured by the time required to process and produce results.

## 7.1 Conclusions

This thesis presented a novel  $\alpha$ -cut representation for type-2 fuzzy sets. This thesis has mainly been centered around this representation and then in an onion-like manner has expanded on this idea.

The main conclusions of this thesis can be summarised as follows:

1. *The  $\alpha$ -cut representation of type-2 fuzzy sets is defined.* It has been proven in Chapter 3 that the  $\alpha$ -cut representation of type-2 fuzzy sets is a mathematical representation that universally preserves the relationship between the different set theoretic formalisms, i.e., type-2 fuzzy sets, interval valued fuzzy sets, standard fuzzy sets and classical sets.
2. *The  $\alpha$ -cut extension principle of type-2 fuzzy sets is defined.* It has been proven in Chapter 3 that the  $\alpha$ -cut representation of type-2 fuzzy sets is functional through an operational calculus for manipulation. The  $\alpha$ -cut extension principle is the basis for this calculus, it can extend operations directly in a simple and elegant manner from classical sets to type-2 fuzzy sets.
3. *The  $\alpha$ -plane extension principle is defined.* It has been proven in Chapter 3 that the  $\alpha$ -plane extension principle can be defined and can extend functions and operations from interval valued fuzzy sets to type-2 fuzzy sets, directly.

4. *The  $\alpha$ -cut representation of type-2 fuzzy sets quantifies uncertainty.* It has been proven in Chapter 4 that the  $\alpha$ -cut extension principle can extend measures of uncertainty to type-2 fuzzy sets.
5. *The  $\alpha$ -cut representation of type-2 fuzzy sets defines arithmetic.* It has been proven in Chapter 5 that the  $\alpha$ -cut extension principle can extend arithmetic operations from classical sets to type-2 fuzzy sets, directly.
6. *Type-2 fuzzy set operations are processed in parallel.* It has been proven in Chapter 6 that the  $\alpha$ -cut representation theorem allow type-2 fuzzy set functions and operations be processed in massively parallel GPUs. Which lead to significant reduction in processing time.

Each of these points will now be explored in greater detail.

### 7.1.1 The Alpha-cut representation of type-2 fuzzy sets is defined

The  $\alpha$ -cut representation for standard fuzzy sets plays a fundamental role in extending functions and operations. These operations are extended through the  $\alpha$ -cut extension principle. The advantage the  $\alpha$ -cut extension has over the original extension principle is that it is set-wise rather than point wise. A comparable representation and extension principle for type-2 fuzzy sets prior to this research work was not available. The  $\alpha$ -cut representation theorem presented in Chapter 3 is a natural extension to the  $\alpha$ -cut representation theorem for fuzzy sets. The main objective of this representation is that it universally preserves the lower order uncertainty semantics across the any uncertainty reduction process. In fact a reduction rule is presented in Chapter 3 that shows that the  $\alpha$ -cut representation reduces to the  $\alpha$ -cut representation of interval valued fuzzy sets when secondary uncertainties about the grade of membership disappears.

This  $\alpha$ -cut representation of interval valued fuzzy sets is also defined in Chapter 3 as an alternative for other attempts that fall short of this reduction rule. If the uncertainty about the membership grade totally diminishes, the  $\alpha$ -cut of a type-2 fuzzy set reduces to the  $\alpha$ -cut definition of a classical fuzzy set. And if all uncertainty about a type-2 fuzzy set disappears it reduces to a crisp defined by the  $\alpha$ -cut at any level. this very important feature of the  $\alpha$ -cut representation for type-2 fuzzy sets makes it universally applicable to all the sets under discussion.

### 7.1.2 The Alpha-cut extension principle of type-2 fuzzy sets is defined

The universality of the  $\alpha$ -cut representation theorem will not be of any use if not for a meaningful calculus for manipulating operations on these sets. This made possible by the  $\alpha$ -cut extension principle for type-2 fuzzy sets presented in Chapter 3. To date no comparable result have been presented for type-2 fuzzy sets. The reduction rule is also satisfied through the  $\alpha$ -cut extension



principle for type-2 fuzzy sets. This particular property makes it very easy to extend the same operation across all the  $\alpha$ -cut extension principles of standard and interval valued fuzzy sets.

The range of functions and relations that can be extended through the  $\alpha$ -cut extension principle is shown to be the same class of functions and relations extended by the  $\alpha$ -cut extension principle of standard fuzzy sets. This cutworthy property allow any function applied to classical fuzzy sets through its  $\alpha$ -cut extension principle can equally be applied to type-2 fuzzy sets.

### **7.1.3 The Alpha-plane extension principle is defined**

The  $\alpha$ -plane representation theorem has recently been proposed as a tool for developing a method of defining the Centroid for a type-2 fuzzy set. In Chapter 3 this representation is used to define an extension principle that allow functions and operations be extended from interval valued fuzzy sets to type-2 fuzzy sets, directly. No comparable results the  $\alpha$ -plane representation theorem have been defined prior to this thesis.

The main value this extension principle add, is the capability to extend operations whether point-valued or set-valued which have been defined for interval valued fuzzy sets to type-2 fuzzy sets. This feature shown to be useful in Chapter 4 to define some of the uncertainty measures for type-2 fuzzy sets.

### **7.1.4 The Alpha-cut representation of type-2 fuzzy sets quantifies uncertainty**

In Chapter 4, uncertainty measures for type-2 fuzzy sets have been defined using the both the  $\alpha$ -plane extension principle and the  $\alpha$ -cut representation theorem for type-2 fuzzy sets. No comparable results using these  $\alpha$ -based extension principles have been defined prior to this thesis. In fact the axioms of these measures have been extended before defining the operations. This was made possible by defining some of the important concepts such as the height, core, support and containment of fuzzy sets in Chapter 3.

The cardinality, subethood and similarity are all considered comparative measures that in fact calculate some features of the type-2 fuzzy set that are useful in some applications. The fuzziness and non-specificity are uncertainty measures that are unique in the standard fuzzy set theory, and widely used in measuring the uncertainty-based information within the set. These measures are significant to the field of generalised information theory, and the methodology presented allow comparative studies between different kinds of sets under study in thesis.

### **7.1.5 The Alpha-cut representation of type-2 fuzzy sets defines arithmetic**

In Chapter 5, type-2 fuzzy numbers and associated arithmetic have been presented. It is a direct implementation of the  $\alpha$ -cut extension principle for type-2 fuzzy sets. Different kinds of interval

valued and type-2 fuzzy numbers have been defined in order to describe different practical situations. It has been shown how to deal with continuous type-2 fuzzy numbers in detail as it needs more attention. It makes use of interval analysis, for example the arithmetic operations can be defined using interval arithmetic. In the discrete case which can directly be implemented a worked example showed how to perform its arithmetic calculations.

The importance of the definition of type-2 fuzzy numbers and arithmetic is that it is a new beginning for a completely new area. Prior to this thesis no formal definition of type-2 fuzzy numbers comparable to that of Chapter 5 is available. The arithmetic operations satisfy the reduction rule, and can be equally used to define operations for standard and interval valued fuzzy sets.

### **7.1.6 Type-2 fuzzy set operations are processed in parallel**

In Chapter 6, the independent nature of the  $\alpha$ -cuts of type-2 fuzzy sets have been investigated. This independent nature allows operations on type-2 fuzzy sets to be processed in parallel. The operation of type-2 systems on massively parallel devices (GPUs) are shown to be successfully applied. It is well known that for a continuous type-2 fuzzy set, the increase on the number of  $\alpha$ -cuts gives a more accurate solution. At the same time the increase in the number of  $\alpha$ -cuts gives increased performance on the GPUs.

The performance gain of the GPU has been proven to always increase as the number of  $\alpha$ -cuts increase. This is shown across all the experiments. The second important conclusion showed that the increase in the number of parallel operations increases the performance gain of the GPU over the CPU. The second experiment showed that the increase in bandwidth and memory speed affects the overall processing time very significantly. This proved that the  $\alpha$ -cut representation is ideal for the current trends in GPU technology development.

The comparison between the  $\alpha$ -cut and  $\alpha$ -plane representations of type-2 fuzzy sets showed that the difference in speed of processing using the GPU is extremely significant. The discussion also demonstrated that performing the addition using the point-wise methods such as the vertical slice representation and the wavy slice representations are very difficult to implement on the GPUs because of their iterative nature. Overall, Chapter 6 demonstrated without further doubt that the  $\alpha$ -cut representation is suitable for parallel processing using GPUs, and allowed for first time the implementation of a full blown type-2 fuzzy set on a GPU. This is expected to be crucial for the future of type-2 fuzzy applications.

## **7.2 Future work**

This thesis has presented the novel  $\alpha$ -cut representation theorem for type-2 fuzzy sets. Although several questions have been answered, but many areas of further work can be explored and needs

more investigation. The following areas of future work can be identified:

- Generalisation** The relation between the different extensions of fuzzy sets has been identified, namely, between type-1 fuzzy sets, interval valued fuzzy sets and type-2 fuzzy sets. This relation has been associated with classical sets through the concept of  $\alpha$ -cuts. These relations may as well be extended to more generalised type-n fuzzy sets and interval type-n fuzzy sets in a progressive manner. Another generalisation may affect other non-standard fuzzy sets such as Intuitionistic fuzzy sets, vague sets, Grey sets, and so on.
- Relationship** The  $\alpha$ -cut representation theorem of type-2 fuzzy sets involves defining a number of crisp set pairs that have close relation to the concept of rough sets. The  $\alpha$ -planes on the other hand have close relation to fuzzy rough sets. This connection between rough sets and type-2 fuzzy sets is definitely worth further investigation. It has the potential of introducing new ideas for both fields.
- Extensions** The concept of a type-2 fuzzy number opens a new area of type-2 fuzzy mathematics. Many mathematical functions can be developed such as the square, square root, logarithm, equations, etc. All of which can be extended directly from classical set theory. The mathematics allow the concepts and functions from the fields of probability for example be extended to type-2 fuzzy probabilities. A range of averaging functions and aggregation operation can be extended such as the Choques integral, which is popular in many applications.
- Applications** The massively parallel nature of the  $\alpha$ -cut representation theorem has the potential of real time applications to be explored. Further work would be to develop real time systems that can make use of GPUs. The implementation of type-2 fuzzy sets on GPUs also introduces new fields of application which GPUs are already powerful such as image processing.

### 7.3 Summary

In this thesis, the foundation for the theory and applications of type-2 fuzzy alpha-cuts is presented. A theory that is expected to have great implications on the progress and advancement of both the theoretical and practical aspects of type-2 fuzzy sets. This theory opens a wide range of opportunities for active research in type-2 fuzzy sets and generalised information theory. Whether this can lead to type-2 fuzzy sets being implemented and applied largely in the industry is an open question. However, with number of functions and operations provided by this dissertation it will definitely lead to more theoretical investigation.

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## Appendix A

# GPU Computing using CUDA C

In the following a primer on the memory management and architecture of the GPUs is provided. GPUs are massively parallel processing units mainly designed for graphics rendering. Only recently GPUs have been used for general purpose computing That span many disciplines. NVIDIA released a line of GPUs that support both graphics rendering and general purpose computing. Their support of GPU computing have culminated by the release of a software toolkit and SDK called CUDA, that made it possible to easily develop applications. Now, CUDA technology is getting more popular amongst academia and the number of publications that are using this technology is growing. A comparison between the architecture of the GPU and CPU can be found in Figure A.1. All the material of this primer is taken from NVIDIA (2010) and Kirk & Wen-me (2010).

- In this thesis the NVIDIA CUDA programming model is used. This model provides extensions to the C programming language in order to make use of NVIDIA GPUs.
- The CUDA model allows C/C++ functions to be executed in parallel on the GPUs, which at the same time give low learning curve. These functions within the CUDA development environment are called *kernels*. These kernels when executed in the GPU are running as *threads* on the multiprocessors provided by the GPU.
- The threads differ in their numbers from one GPU to the other. They are arranged in a hierarchy such that there are a fixed number of threads that form a *block* of threads. These blocks are also arranged independently, and there are fixed number of blocks that form a *grid*.
- The independence of these blocks are shown through their memory, which can only be accessed by the threads of that particular block. The block memory is relatively small and only shared by the threads in that block only, which makes it as fast as register access.
- There is also a global memory for the whole GPU which is relatively large. This memory is

accessible by all threads of all the blocks. The problem with this memory is that it has very high latency.

- Another type of memory available for each thread individually and not shared by any other is called the local memory. This local memory also has high latency, but there is also a constant memory available for all threads. This constant memory is readable for all the threads with relatively fast access and low latency.
- Figure A.2 shows a schematic diagram of the CUDA programming model. It is very important to stress on the issue of memory access optimisation. The memory should be accessed in such a way that the kernels will make the best use of the limited shared memory.
- It is obvious that the best practice is to send the data in full and then send the instruction to process the data. Iterations and recursive algorithms cause delay to the GPU processing time because of their high demand for shared memory.



Fig. A.1. GPUs compared to CPUs, taken from Nvidia (2010).

From the year 2010 some textbooks and simplified books for CUDA programming has been published. The number of papers that involve CUDA programming is growing significantly. This powerful tool for general purpose computing is proving to be crucial for future processing.

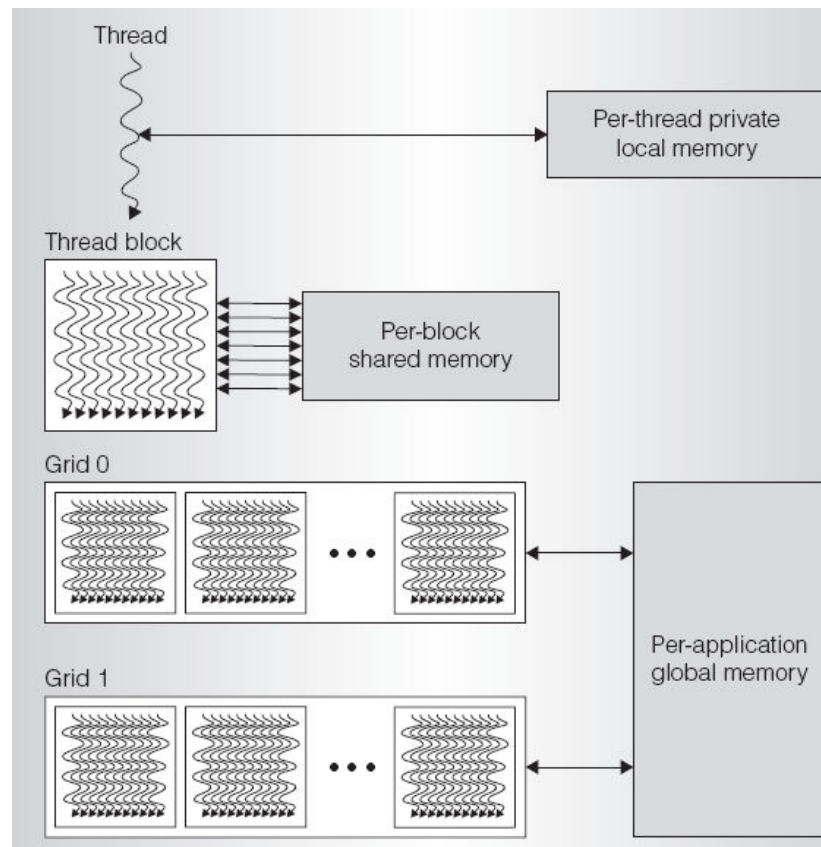


Fig. A.2. CUDA programming model, taken from Kirk and Wen-mei (2010).



## Appendix B

### List of publications by the author

The following is a list of publications by the author which is related to the thesis:

- Hamrawi, H., Coupland, S. and John, R. (2009), Extending operations on type-2 fuzzy sets, in Computational Intelligence (UKCI), 2009 UK Workshop on, Essex, UK.
- Hamrawi, H. and Coupland, S. (2009a), Non-specificity measures for type-2 fuzzy sets, in Fuzzy Systems, 2009. FUZZ-IEEE 2009. IEEE International Conference on, Korea, pp. 732737.
- Hamrawi, H. and Coupland, S. (2009b), Type-2 fuzzy arithmetic using alpha-planes, in Fuzzy Systems, 2009. IFSA/EUSFLAT 2009. IFSA International Congress on, Portugal, pp. 606611.
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